Monday: Subsequences and monotonic sequences

Subsequences [5.5]

A *subsequence* of a sequence (s_n) is a sequence formed by taking certain terms from the original sequence, in the same order as they appeared in the original sequence. For example, if we have the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ then we may form the subsequence $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$. More precisely, we have the following definition.

Definition. *A* subsequence *of a sequence (s_n) is a sequence (s_{i_n}), where (i_n) is a strictly increasing sequence in $\mathbb{N}$.*

Lemma 1. *If (i_n) is a strictly increasing sequence in $\mathbb{N}$ then for all $n \leq i_n$, $n \leq i_n$.*

Proof. Exercise (Assignment 5, Question 5). □

Proposition 2. *Let (s_n) be a sequence in $\mathbb{R}$, and (s_{i_n}) a subsequence of (s_n). If $s_n \to L$ as $n \to \infty$ then $s_{i_n} \to L$ as $n \to \infty$.*

Proof. Suppose $s_n \to L$ as $n \to \infty$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that if $n > N$ then $|s_n - L| < \varepsilon$. Now let $n > N$. Then $i_n \geq n > N$, so $i_n > N$, so $|s_{i_n} - L| < \varepsilon$. □

Theorem 3. *Let (s_n) be a monotonic bounded sequence in $\mathbb{R}$. Then (s_n) converges to some $L \in \mathbb{R}$*

Proof. Suppose first that (s_n) is increasing. The set $S = \{\, s_n : n \in \mathbb{N} \,\}$ is non-empty (since $s_1 \in S$) and bounded above, so it has a least upper bound, L say. We claim that $s_n \to L$ as $n \to \infty$. So let $\varepsilon > 0$. Then there is some $s \in S$ with $L - \varepsilon < s \leq L$. Now, $s \in S$ so $s = s_N$ for some $N \in \mathbb{N}$. Let $n > N$. Then $s_N \leq s_n$, since (s_n) is increasing, so we have $L - \varepsilon < s_N \leq s_n \leq L < L + \varepsilon$, so $L - \varepsilon < s_n < L + \varepsilon$, so $|s_n - L| < \varepsilon$. Thus $s_n \to L$, as claimed.

We leave the case when (s_n) is a decreasing sequence as an exercise. □

Theorem 4. *Let (s_n) be a sequence in $\mathbb{R}$. Then (s_n) has a subsequence which is monotonic.*

The idea is as follows: we give a method for constructing an increasing subsequence in (s_n), which will work unless some particular thing goes wrong. We will then give an alternative method which gives a decreasing subsequence, and which will work if that particular thing went wrong with the first method.

Lemma 5. *Let (s_n) be a sequence in $\mathbb{R}$ with no greatest term. Then (s_n) has an increasing subsequence.*

Proof. We construct the subsequence (s_{i_n}) recursively. The sequence has the property that

$$\text{for all } j,k \in \mathbb{N}, \text{ if } j \leq i_k \text{ then } s_j \leq s_{i_k}. \tag{$*$}$$

First we let $i_1 = 1$. This certainly satisfies $(*)$ since there is no j with $j < 1$. Now suppose we have chosen $i_1 < i_2 < \cdots < i_n$ satisfying $(*)$. We know that s_{i_n} is not the greatest term in the sequence, since there is no greatest term, so there is some m with $s_{i_n} < s_m$. However, $s_j \leq s_{i_n}$ for all $j \leq i_n$, so if $s_{i_n} < s_m$ then $m > i_n$. We let i_{n+1} be the least $m > i_n$ with $s_{i_n} \leq s_m$. We must check that this choice also satisfies $(*)$. We have assumed that it is satisfied for all i_ks for $k \leq n$, so we only need to check it for i_{n+1}. So suppose $j < i_{n+1}$. If $j \leq i_n$ then $s_j \leq s_{i_n} \leq s_{i_{n+1}}$. If $i_n < j < i_{n+1}$ then, since i_{n+1} was the least m with $s_{i_n} \leq s_m$, we must have $s_j < s_{i_n} \leq s_{i_{n+1}}$.

Clearly, the subsequence (s_{i_n}) we have constructed is an increasing sequence, as required. □

Proof of Theorem 4. Let (s_n) be a sequence in $\mathbb{R}$. There are two possibilities: either there is an $n \in \mathbb{N}$ such that $\{ s_m : m > n \}$ has no greatest element, or there is no such n. In the latter case, for every $n \in \mathbb{N}$, $\{ s_m : m > n \}$ has a greatest element.

Case 1: Suppose there is some n_0 such that $\{ s_m : m > n_0 \}$ has no greatest element. For each k, put $t_k = s_{n_0+k}$. Then (t_k) has no greatest element, so by the previous lemma it has an increasing subsequence (t_{i_k}). But then $(s_{n_0+i_k})$ is an increasing subsequence of (s_n).

Case 2: Suppose that for every $n \in \mathbb{N}$, $\{ s_m : m > n \}$ has a greatest element. Recursively choose a subsequence of (s_n) as follows: i_1 is chosen so that $s_{i_1} \geq s_m$ for all $m > 1$, and once $i_1 < i_2 < \cdots < i_n$ have been chosen, i_{n+1} is chosen so that $i_n < i_{n+1}$ and $s_{i_{n+1}} \geq s_m$ for all $m > n$. Since $\{ s_m : m > n \}$ always has a greatest element, we can always find such i_1 and i_{n+1}. It remains only to show that this gives a decreasing subsequence. Note that for each n we have that s_{i_n} is the greatest element of $\{ s_m : m > k \}$ for some $k < i_n$, so $s_{i_n} \geq s_m$ for all $m > k$. In particular, since $k < i_n < i_{n+1}$, $s_{i_n} \geq s_{i_{n+1}}$ as required.

□

Tuesday: Cauchy sequences

We know what it means to say that (s_n) converges to L. To say that (s_n) converges means that (s_n) converges to some L, i.e.

$$(\exists L)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)(|s_n - L| < \varepsilon).$$

This is rather complicated: it has an extra layer of complexity with the extra change between $\exists$ and $\forall$ quantifiers. It is also awkward to check, since we have to find the limit L before we can check that the condition holds. An alternative property, which only mentions the sequence itself and not its possible limit, is the "Cauchy convergence criterion":

Definition. *A sequence (s_n) in $\mathbb{R}$ is a* Cauchy *sequence if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $|s_m - s_n| < \varepsilon$.*

We will prove that a sequence (s_n) in $\mathbb{R}$ converges if and only if it is a Cauchy sequence.

Lemma 6 (The Triangle Inequality). *Let $a, b \in \mathbb{R}$. Then $|a+b| \le |a| + |b|$, and hence, if $x, y, z \in \mathbb{R}$ then $|x-z| \le |x-y| + |y-z|$.*

Proof. Exercise. □

Proposition 7. *Let (s_n) be a sequence in $\mathbb{R}$. If (s_n) converges then (s_n) is bounded.*

Proof. Suppose $s_n \to L$ as $n \to \infty$. Putting $\varepsilon = \frac{1}{2}$, we know that there is some $N \in \mathbb{N}$ such that if $n > N$ then $|s_n - L| < \frac{1}{2}$. So, for $n > N$ we have

$$|s_n| = |(s_n - L) + L| \le |s_n - L| + |L| < |L| + \tfrac{1}{2}.$$

Thus for every n we have $|s_n| \le \max\{|s_1|, |s_2|, \ldots, |s_N|, |L| + \frac{1}{2}\}$. So (s_n) is bounded. □

Lemma 8. *Let (s_n) be a bounded sequence. Then (s_n) has a convergent subsequence.*

Proof. We know that any sequence in $\mathbb{R}$ has a monotonic subsequence, and any subsequence of a bounded sequence is clearly bounded, so (s_n) has a bounded monotonic subsequence. But every bounded monotonic sequence converges. So (s_n) has a convergent subsequence, as required. □

Lemma 9. *Let (s_n) be a Cauchy sequence in $\mathbb{R}$. If (s_n) has a convergent subsequence then (s_n) converges.*

Proof. Let (s_{i_n}) be a subsequence which converges to L. Let $\varepsilon > 0$. Put $\eta = \varepsilon/2$. Choose N_1 so that if $m, n > N_1$ then $|s_m - s_n| < \eta$, choose N_2 so that if $n > N_2$ then $|s_{i_n} - L| < \eta$, and choose k so that $k > N_2$ and $i_k > N_1$ (for example, we may take $k = \max\{N_1 + 1, N_2 + 1\}$: certainly $k > N_2$ and $i_k \ge k > N_1$). Put $N = N_1$. Then

$$\begin{aligned}
|s_n - L| &= |s_n - s_{i_k} + s_{i_k} - L| \\
&\le |s_n - s_{i_k}| + |s_{i_k} - L| && \text{(triangle inequality)} \\
&< \eta + |s_{i_k} - L| && \text{(since } n, i_k > N_1) \\
&< \eta + \eta && \text{(since } k > N_2) \\
&= \varepsilon.
\end{aligned}$$

Thus $|s_n - L| < \varepsilon$ as required. So (s_n) converges to L. □

Lemma 10. *Every Cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Exercise. □

Lemma 11. *Every convergent sequence in $\mathbb{R}$ is Cauchy.*

Proof. Exercise. □

Putting these results together gives our main result:

Theorem 12. *A sequence in $\mathbb{R}$ is a Cauchy sequence if and only if it converges.*

Limits of sums and products

Theorem 13. *Let (a_n), (b_n) be sequences in $\mathbb{R}$. Suppose that $a_n \to A$ and $b_n \to B$ as $n \to \infty$. Then*

1. $a_n + b_n \to A + B$ *as* $n \to \infty$*;*
2. $a_n b_n \to AB$ *as* $n \to \infty$*; and*
3. *if* $b_n \neq 0$ *for all* n *and* $B \neq 0$ *then* $\frac{a_n}{b_n} \to \frac{A}{B}$ *as* $n \to \infty$*.*

Proof. For (1), let $\varepsilon > 0$. Put $\eta = \varepsilon/2$. Choose $N_1, N_2 \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - A| < \eta$ and if $n > N_2$ then $|b_n - B| < \eta$. Put $N = \max\{N_1, N_2\}$. Let $n > N$. Then

$$\begin{aligned} |(a_n + b_n) - (A + B)| &= |(a_n - A) + (b_n - B)| \\ &\leq |a_n - A| + |b_n - B| && \text{(triangle inequality)} \\ &< \eta + \eta && \text{(since } n > N_1 \text{ and } n > N_2\text{)} \\ &= \varepsilon. \end{aligned}$$

so $a_n + b_n \to A + B$ as $n \to \infty$.

For (2), let $\varepsilon > 0$. Since (b_n) converges, it is bounded, so there is some $P > 0$ with $|b_n| < P$ for all n. Put $\eta = \frac{\varepsilon}{|A|+P}$. Choose $N_1, N_2 \in \mathbb{N}$ such that if $n > N_1$ then $|a_n - A| < \eta$ and if $n > N_2$ then $|b_n - B| < \eta$. Put $N = \max\{N_1, N_2\}$. Let $n > N$. Then

$$\begin{aligned} |a_n b_n - AB| &= |a_n b_n - A b_n + A b_n - AB| \\ &\leq |a_n b_n - A b_n| + |A b_n - AB| && \text{(triangle inequality)} \\ &= |a_n - A||b_n| + |A||b_n - B| \\ &= |a_n - A|P + |A||b_n - B| \\ &< \eta P + |A|\eta \\ &= \varepsilon \end{aligned}$$

Thus $a_n b_n \to AB$ as $n \to \infty$.

For (3), we will first prove that $\frac{1}{b_n} \to \frac{1}{B}$ and then apply 2. So let $\varepsilon > 0$. Put $\eta = \frac{|B|^2\varepsilon}{2}$. Since $B \neq 0$, $\frac{|B|}{2} > 0$, so there is some N_1 such that if $n > N_1$ then $|b_n - B| < \frac{|B|}{2}$. Note that if $n > N_1$ then $|b_n| > |B| - \frac{|B|}{2} = \frac{|B|}{2}$, so $\left|\frac{1}{b_n}\right| < \frac{2}{|B|}$. Choose $N_2 \in \mathbb{N}$ such that if $n > N_2$ then $|b_n - B| < \eta$. Put

$N = \max\{N_1, N_2\}$. Let $n > N$. Then

$$\begin{aligned}\left|\frac{1}{b_n} - \frac{1}{B}\right| &= \left|\frac{B - b_n}{b_n B}\right| \\ &= \left|\frac{1}{b_n}\right|\left|\frac{1}{B}\right||B - b_n| \\ &< \frac{2}{|B|}\frac{1}{|B|}|b_n - B| \\ &< \frac{2}{|B|^2}\eta \\ &= \varepsilon,\end{aligned}$$

so $\frac{1}{b_n} \to \frac{1}{B}$ as $n \to \infty$. The result then follows by (2). □

Thursday: Continuous functions

Definition. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, and let $a \in A$. Then f is* continuous at a *if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in A$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. We say that f is* continuous *if it is continuous at a for all $a \in A$.*

Example 14. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$, and let $a \in \mathbb{R}$. Then f is continuous at a.*

Proof. Let $\varepsilon > 0$. Put $\delta = \min\{1, \frac{\varepsilon}{2|a|+1}\}$. Let $x \in \mathbb{R}$ with $|x - a| < \delta$. Put $h = x - a$, so $x = a + x$. Then

$$\begin{aligned}|f(x) - f(a)| &= |f(a + h) - f(a)| \\ &= |(a + h)^2 - a^2| \\ &= |a^2 + 2ah + h^2 - a^2| \\ &= |2ah + h^2| \\ &= |2a + h||h| \\ &\le (|2a| + |h|)|h| \\ &\le (2|a| + 1)|h| && \text{(since } |h| < 1) \\ &< (2|a| + 1)\delta \\ &= \varepsilon,\end{aligned}$$

as required. □

Example 15. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$. Then f is not continuous at 0.*

Proof. Suppose for a contradiction that f is continuous at 0. Then, since $\frac{1}{2} > 0$, there is some $\delta > 0$ such that if $|x - 0| < \delta$ then $|f(x) - f(0)| < \frac{1}{2}$. Choose $n \in \mathbb{N}$ with $n > \frac{1}{2}\left(\frac{2}{\pi\delta} - 1\right)$. Then $2n + 1 > \frac{2}{\pi\delta}$, so $\frac{(2n+1)\pi}{2} > \frac{1}{\delta}$, so $\frac{2}{(2n+1)\pi} < \delta$. Put $x = \frac{2}{(2n+1)\pi}$. Then $|x| < \delta$, so $|f(x)| < \frac{1}{2}$. However, $f(x) = \sin\left((2n + 1)\frac{\pi}{2}\right)$, so $f(x) = \pm 1$, so $|f(x)| = 1 \not< \frac{1}{2}$. This contradiction shows that there is no such δ, and hence f is not continuous at 0. □

The intermediate value theorem

Theorem 16 (The intermediate value theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $k \in \mathbb{R}$ with $f(a) < k < f(b)$. Then there is some $c \in (a, b)$ with $f(c) = k$.*

Proof. Put $S = \{\, x \in [a, b] : f(x) < k \,\}$. Then $a \in S$ so $S \neq \emptyset$, and S is bounded above by b, so S has a supremum. Put $c = \sup S$.

Claim: $f(c) \not< k$.

For: Suppose for a contradiction that $f(c) < k$. Put $\varepsilon = k - f(c)$, and choose $\delta > 0$ so that if $x \in [a, b]$ with $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Note that if $|f(x) - f(c)| < \varepsilon$ then $f(x) - f(c) < \varepsilon = k - f(c)$, so $f(x) < f(c)$. Thus $|b - c| \not< \delta$, so $c + \delta \leq b$. Put $x = c + \frac{\delta}{2}$. Then $x > c = \sup S$, so $x \notin S$. However, $f(x) < f(c) + \varepsilon = k$, and $x \in [a, b]$, so $x \in S$. This contradiction showws that we cannot have $f(c) < k$.

Claim: $f(c) \not> k$.

For: Suppose for a contradiction that $f(c) > k$. Put $\varepsilon = f(c) - k$. Choose $\delta > 0$ such that if $x \in [a, b]$ with $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. Since $\delta > 0$ and $c = \sup S$, there is some $x \in S$ with $c - \delta < x \leq c$. But then $|x - c| < \delta$, so $|f(x) - f(c)| < \varepsilon$, so $f(x) - f(c) > -\varepsilon = -(f(c) - k) = k - f(c)$. Thus $f(x) > k$. But this contradicts the assumption that $x \in S$ so $f(x) < k$. Hence there is no such x and therefore we cannot have $f(c) > k$.

Thus we cannot have $f(c) < k$ or $f(c) > k$, so $f(c) = k$, as required. Finally, note that since $a \in S$ and b is an upper bound for S, $a \leq supS \leq b$, i.e. $a \leq c \leq b$. Since $f(a) \neq f(c) \neq f(b)$ we have $a \neq c \neq b$ so $a < c < b$, i.e. $c \in (a, b)$ as required. □

Friday: Continuity in terms of limits, open and closed sets and sequences

Limits of functions

Definition. *Let $a \in \mathbb{R}$ and let $\varepsilon > 0$. We define the* ε-ball centred at a, *$B_\varepsilon(a)$, by*

$$B_\varepsilon(a) = \{\, x \in \mathbb{R} : |x - a| < \varepsilon \,\},$$

and the deleted ε-ball centred at a, *$B'_\varepsilon(a)$, by $B'_\varepsilon(a) = B_\varepsilon(a) \setminus \{a\}$.*

Definition. *Let $A \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$. Then a is a* limit point *of A if, for every $\varepsilon > 0$, $B_\varepsilon(a) \cap A \neq \emptyset$, and a is an* accumulation point *of A if for all $\varepsilon > 0$, $B'_\varepsilon(a) \cap A \neq \emptyset$.*

Definition. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, let a be an accumulation point of A and let $L \in \mathbb{R}$. We say that $\lim_{x \to a} f(x) = L$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in A$, if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.*

Notice the big difference between the definition of a limit and the definition of continuity: we insist that $0 < |x - a| < \delta$, in other words we do not test whether $|f(x) - L| < \varepsilon$ holds at $x = a$, only at values of x close to but not exactly equal to a. Thus, for example $\lim_{x\to 0} \frac{\sin x}{x}$ makes sense without having to explain that we never intend to evaluate $\frac{\sin 0}{0}$.

Example 17. *Define the function* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x$ *if* $x \notin \mathbb{Z}$, $f(x) = 0$ *if* $x \in \mathbb{Z}$. *Then* $\lim_{x\to 2} f(x) = 2$.

The two definitions, continuity and limits, fit together by the following result.

Theorem 18. *Let* $A \subseteq \mathbb{R}$ *and let* $f : A \to \mathbb{R}$ *be a function. Then* f *is continuous if and only if, for every* $a \in A$*, if* a *is an accumulation point of* A *then* $\lim_{x\to A} f(x) = f(a)$.

Proof. Exercise. □

Open and closed sets

Definition. *A subset* U *of* $\mathbb{R}$ *is* open *if for every* $x \in U$ *there is some* $\varepsilon > 0$ *such that* $B_\varepsilon(x) \subseteq U$. *A subset* C *of* $\mathbb{R}$ *is* closed *if* $C^{\complement}_{\mathbb{R}}$ *is open.*

Proposition 19. *Let* $C \subseteq \mathbb{R}$. *Then* C *is closed if and only if, for every sequence* (s_n) *in* C*, if* $(s_n) \to a$ *as* $n \to \infty$ *then* $a \in C$.

Proof. Suppose first that C is closed. We must show that if (s_n) is a convergent sequence in C then the limit of the sequence is also in C. So suppose that $s_n \to a$ as $n \to \infty$. Suppose, for a contradiction that $a \notin C$. Then $a \in C^{\complement}$, and $C^{\complement}$ is open, so there is an $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq C^{\complement}$. Since $s_n \to a$, there is an $N \in \mathbb{N}$ such that for $n > N$, $|s_n - a| < \varepsilon$. But then $s_{N+1} \in B_\varepsilon(a)$, so $s_{N+1} \in C^{\complement}$, contradicting the assumption that (s_n) is a sequence in C. So we cannot have $a \notin C$, so $a \in C$.

Conversely, suppose that for every sequence in C, if $s_n \to a$ then $a \in C$. Put $U = C^{\complement}$. We must show that U is open. So let $a \in U$. Suppose, for a contradiction, that there is no $\varepsilon > 0$ with $B_\varepsilon(a) \subseteq U$. In particular, for each $n \in \mathbb{N}$ we have $B_{\frac{1}{n}}(a) \nsubseteq U$, so there is some $s_n \in B_{\frac{1}{n}}(a) \setminus U$. But then $s_n \notin C^{\complement}$, so $s_n \in C$.

Claim: $s_n \to a$ as $n \to \infty$.

For: Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. Then $\frac{1}{N} < \varepsilon$. Let $n \in \mathbb{N}$ with $n > N$. Then $\frac{1}{n} < \frac{1}{N}$. Since $s_n \in B_{\frac{1}{n}}(a)$, $|s_n - a| < \frac{1}{n} < \frac{1}{N} < \varepsilon$, so $|s_n - a| < \varepsilon$ as required.

Thus (s_n) is a sequence in C which converges to a, but $a \notin C$, contradicting our assumption about C. □

Lemma 20. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. For every open set* U*,* $f^{-1}(U)$ *is open.*

Proof. Let U be open, and let $a \in f^{-1}(U)$. Then $f(a) \in U$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq U$. By continuity, there is some $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Claim: $B_\delta(a) \subseteq f^{-1}(U)$.

For: Let $x \in B_\delta(a)$. Then $|x - a| < \delta$, so $|f(x) - f(a)| < \varepsilon$, so $f(x) \in B_\varepsilon(f(a)) \subseteq U$, so $f(x) \in U$, so $x \in f^{-1}(U)$, as required.

□

The converse is also true: to prove it, we first have to use the triangle inequality to prove that every ε-ball $B_\varepsilon(a)$ is open.

Lemma 21. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is continuous if and only if, for every sequence (s_n) in $\mathbb{R}$, if $s_n \to a$ as $n \to \infty$ then $f(s_n) \to f(a)$ as $n \to \infty$.*

Proof. Suppose first that f is continuous. Let (s_n) be a sequence in $\mathbb{R}$. Suppose $s_n \to a$ as $n \to \infty$. Let $\varepsilon > 0$. By continuity, there is some $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. Since $s_n \to a$, there is some N such that if $n > N$ then $|s_n - a| < \delta$. Let $n > N$. Then $|s_n - a| < \delta$, so $|f(s_n) - f(a)| < \varepsilon$, as required.

We leave the converse as an exercise. □