Monday: The Real Numbers

In this final section of the course we will study the real numbers. You are already familiar with a number of theorems about the real numbers: rules for n^{th} root test for determining whether a series converges, the mean value theorem, and so on. We will be learning how to prove theorems like these.

Of course, the first thing we have to do is to establish our assumptions, or *axioms*, and agree that (at least in principle) everything we prove about the real numbers should come **only** from these axioms and not from any pictures we have of how the real numbers look and behave. We will give axioms which say that the real numbers are a *complete, ordered field*. There are lots of examples of fields, some of which are ordered fields. However, we will see that there is, up to isomorphism, only one complete ordered field. By "up to isomorphism", we mean that if R and S are both complete ordered fields, then there is an isomorphism from R to S. In fact, in this case we can do even better: not only is there at least one isomorphism from R to S, but that isomorphism is unique.

The field axioms [8.2]

Definition. *A* field *is a set F equipped with two binary operations, addition $+$ and multiplication $\cdot$ (as usual, we often omit the $\cdot$ and write $x \cdot y$ as xy) and distinct elements 0_F and 1_F with the properties that*

- *$+$ and $\cdot$ are associative and commutative operations on F;*
- *0_F is an identity for $+$ and 1_F is an identity for $\cdot$;*
- *$\cdot$ distributes over $+$, i.e. for all $x, y, z \in F$ we have $x(y + z) = xy + xz$;*
- *every $x \in F$ has an additive inverse $-x$; and*
- *every $x \in F \setminus \{0_F\}$ has a multiplicative inverse $\frac{1}{x}$.*

Example 1. *The real numbers $\mathbb{R}$, with the usual addition, multiplication, 0 and 1, form a field.*

Example 2. *Let $F = \{E, O\}$, with $+$ and $\cdot$ defined by the Cayley Tables*

$+$	E	O
E	E	O
O	O	E

and

$\cdot$	E	O
E	E	E
O	E	O

(Note: it may help to think of E and O as "even" and "odd" respectively). Then F is a field, with 0_F being the element E and 1_F being the element O.

Exercise 3. *In the previous example, what are* $-E$, $-O$ *and* $\frac{1}{O}$*?*

Example 4. *Let* p *be a prime number. Then* $\mathbb{Z}_p$ *is a field, with* $+$ *and* $\cdot$ *being* $+_p$ *and* $\cdot_p$ *respectively, and* 0_F *and* 1_F *beling* $\overline{0}$ *and* $\overline{1}$ *respectively.*

Exercise 5. *Write up the Cayley Tables for* $+_7$ *and* $\cdot_7$*. For each* n *with* $0 \leq n \leq 6$ *identify* $-\overline{n}$ *and for each* n *with* $1 \leq n \leq 6$ *identify* $\frac{1}{\overline{n}}$*.*

Just as when we wrote down axioms for $\mathbb{Z}$ and $\mathbb{N}$, we can deduce many familiar facts about a field from these axioms.

Proposition 6. *Let* F *be a field. For every* $x \in F$ *we have* $0_F x = 0_F$ *and* $-(-x) = x$*.*

However, the field axioms do **not** allow us to prove the familiar fact that for all $x \neq 0_F$ we have $x \neq -x$.

Finally, a little notation: if F is a field and $x, y \in F$, we write $x - y$ for $x + (-y)$, and $x \div y$ or $\frac{x}{y}$ for $x \cdot \frac{1}{y}$.

Tuesday: The Order Axioms for $\mathbb{R}$

Axioms for an ordered field [8.3]

Definition. *An* ordered field *is a field* F *with a subset* P *such that*

- *if* $x, y \in P$ *then* $x + y \in P$ *and* $xy \in P$*;*
- *for all* $x \in F$*, exactly one of the following holds:*
  - $x \in P$*; or*
  - $x = 0_F$*; or*
  - $-x \in P$*.*

We will see shortly why we use the term "ordered" (and why we use the letter P). First, we will see some consequences of these axioms.

Proposition 7. *Let* F *be an ordered field. Then for all* $x \in F \setminus \{0_F\}$*,* $x \neq -x$*.*

Proposition 8. *Let* F *be an ordered field. Then for all* $x \in F \setminus \{0_F\}$*,* $x^2 \in P$*. In particular,* $1_F \in P$*.*

Proposition 9. *The field* $\mathbb{Z}_p$ *(for* p *a prime number) is not an ordered field, in other words there is no subset* P *of* $\mathbb{Z}_p$ *which satisfies the ordered field axioms.*

Example 10. *The real numbers are an ordered field, with* $P = \{ x \in \mathbb{R} : x > 0 \}$*.*

Proposition 11. *Let F be an ordered field. Define relations $<$ and $\leq$ on F by declaring that, for $x, y \in F$, $x < y$ iff $y - x \in P$, and $x \leq y$ iff $x < y \vee x = y$. Then $\leq$ is a total order on F.*

Whenever we have an ordered field F, we will always assume $<$ and $\leq$ are the relations defined in the above proposition.

Proposition 12. *Let F be an ordered field. Let $a, b, c, d \in F$.*

1. *If $a < b$ then $a + c < b + c$.*
2. *If $a \leq b$ then $a + c \leq b + c$.*
3. *If $a < b$ and $0_F < c$ then $ac < bc$.*
4. *If $0_F < a < b$ then $0_F < \frac{1}{b} < \frac{1}{a}$.*

Example 13. *The rational numbers $\mathbb{Q}$ are an ordered field, with the usual $+$, $\cdot$, 0 and 1, and with $P = \{ q \in \mathbb{Q} : q > 0 \}$.*

Thursday: Completeness

The ordered field axioms are not yet enough to characterise the real numbers, as there are other examples of ordered fields besides the real numbers. The most familiar of these is the set of rational numbers. The final axiom we give is the completeness axiom, which is satisfied by $\mathbb{R}$ but not by $\mathbb{Q}$.

Definition. *A* complete *ordered field is an ordered field F with the least upper bound property (in other words, with the property that if $S \subseteq F$, $S \neq \emptyset$ and S is bounded above then S has a least upper bound* $\sup S$*).*

Example 14. *The real numbers are a complete ordered field.*

We will see in a moment that the rational numbers are not complete.

Lemma 15. *Let F be a complete ordered field, and let $S \subseteq F$, $x \in F$. Then TFAE:*

- $x = \sup S$
- *x is an upper bound for S and, for each $\varepsilon \in F$ with $\varepsilon > 0_F$ there is some $s \in S$ with $x - \varepsilon < s \leq x$.*

Proposition 16 (The Archimedean property of $\mathbb{R}$). *For every $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ with $n > x$.*

Proof. Let $x \in \mathbb{R}$. Suppose, for a contradiction, that there is no $n \in \mathbb{N}$ with $n > x$. Then, since $\leq$ is a total order, we have $n \leq x$ for all $n \in \mathbb{N}$. Thus $\mathbb{N}$ is bounded above. We also have $\mathbb{N} \neq \emptyset$, so $\mathbb{N}$ must have a least upper bound, s. Since $s = \sup \mathbb{N}$ and $1 > 0$, there is some $n \in \mathbb{N}$ with $s - 1 < n \leq s$. But then $s < n + 1$, so $n + 1 \not\leq s$. However, $n + 1 \in \mathbb{N}$ and s is an upper bound for $\mathbb{N}$ so $n + 1 \leq s$. This contradiction shows that there must be some $n \in \mathbb{N}$ with $n > x$, as required. □

Proposition 17. *There is some real number a with $a^2 = 2$.*

Proof. Let $S = \{ x \in \mathbb{R} : x^2 < 2 \}$.

Claim: $S \neq \emptyset$.

For: $0^2 = 0 < 2$, so $0 \in S$.

Claim: S is bounded above.

For: We will show that 2 is an upper bound for S. So, let $x \in S$. Suppose, for a contradiction, that $x \nleq 2$. Then $x > 2$, so $x^2 > 2^2 = 4 > 2$, so $x^2 \not< 2$, contradicting the assumption that $x \in S$.

From this we know that S has a least upper bound. Put $a = \sup S$: we will show that $a^2 = 2$ as required.

Claim: $a^2 \not< 2$.

For: Suppose, for a contradiction, that $a^2 < 2$. Put $p = 2 - a^2$ and $q = \frac{p}{5}$. Notice that $a \leq 2$, because 2 is an upper bound for S, and $0 < p \leq 2$ so $q < 1$

$$\begin{aligned} (a+q)^2 &= a^2 + 2aq + q^2 \\ &< a^2 + 2 \cdot 2 \cdot q + 1 \cdot q && \text{(since } a \leq 2 \text{ and } q < 1) \\ &= a^2 + 5q \\ &= 2. \end{aligned}$$

So we have $(a+q)^2 < 2$, so $a + q \in S$. But $a < a + q$, contradicting the fact that a is an upper bound for S. Thus we cannot have $a^2 < 2$.

Claim: $a^2 \not> 2$.

For: Suppose, for a contradiction, that $a^2 > 2$. Put $r = a^2 - 2$, and $\varepsilon = \frac{r}{2}$. Then $\varepsilon > 0$, so since $a = \sup S$ there is some $s \in S$ with $a - \varepsilon < s \leq a$. Since $s > a - \varepsilon$ we have

$$\begin{aligned} s^2 &> (a - \varepsilon)^2 \\ &= a^2 - 2\varepsilon + \varepsilon^2 \\ &\geq a^2 - 2\varepsilon && \text{(since } \varepsilon^2 \geq 0) \\ &= 2, \end{aligned}$$

so $s^2 > 2$, contradicting the assumption that $s \in S$. This shows that we cannot have $a^2 > 2$.

Hence we must have $a^2 = 2$, as required. □

Friday: Sequences

Sequences [5.5, 8.5]

Definition. *Let A be a set. A* sequence in A *is a function $s : \mathbb{N} \to A$. We usually write $s(n)$ as s_n, and we write (s_n) or $s_1, s_2, s_3, \ldots$ for the whole sequence.*

Example 18. *The sequence $(\frac{n-1}{n})$ has $s_n = \frac{n-1}{n}$, so it is the sequence $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots$.*

Definition. *Let (s_n) be a sequence in $\mathbb{R}$. We say that (s_n) is*

- increasing *if for all $n \in \mathbb{N}$, $s_n \leq s_{n+1}$;*
- strictly increasing *if for all $n \in \mathbb{N}$, $s_n < s_{n+1}$;*
- decreasing *if for all $n \in \mathbb{N}$, $s_n \geq s_{n+1}$;*
- strictly decreasing *if for all $n \in \mathbb{N}$, $s_n > s_{n+1}$;*
- monotonic *if it is either increasing or decreasing;*
- bounded above *if $\{ s_n : n \in \mathbb{N} \}$ is bounded above;*
- bounded below *if $\{ s_n : n \in \mathbb{N} \}$ is bounded below; and*
- bounded *if it is both bounded above and below.*

Example 19. *The sequence $(\frac{n-1}{n})$ is strictly increasing (si it is increasing, so it is monotone), and is bounded above by 1 and below by 0, so it is bounded.*

Definition. *For $a \in \mathbb{R}$ we define $|a|$ by*

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{otherwise.} \end{cases}$$

Thus we have $|a| \geq 0$ for all $a \in \mathbb{R}$, with $|a| > 0$ unless $a = 0$.

Proposition 20. *For any $a, x \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, we have $|a - x| < \varepsilon$ iff $a - \varepsilon < x < a + \varepsilon$.*

Proof. Exercise. □

Definition. *Let (s_n) be a sequence in $\mathbb{R}$, and let $L \in \mathbb{R}$. We say that (s_n)* converges to L *if for every $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - L| < \varepsilon$. If (s_n) converges to L, we write $s_n \to L$ as $n \to \infty$, and call L a* limit *of the sequence (s_n).*

Example 21. *The sequence $(\frac{n-1}{n})$ converges to 1.*

Example 22. *The sequence $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$ converges to 0.*

Theorem 23. *If the sequence (s_n) in $\mathbb{R}$ has a limit, then the limit is unique.*