1. (a) (**3 marks**) "n is even and n^2 is odd or $n+1$ is even".

(b) (**2 marks**) "If n^2 is odd or $n+1$ is even, then n is odd".

(c) (**2 marks**) "If n^2 is even and $n+1$ is odd, then n is even".

(d) (**3 marks**) It is true. Let $n \in \mathbb{N}$ be even, so that $n = 2k$ for some $k \in \mathbb{N}$. Thus $n^2 = (2n)^2 = 2(2n^2)$ and $n+1 = 2k+1$. So n^2 is even and $n+1$ is odd.

2. (a) (**7 marks**) Let $P(n)$ be the statement "$4^{3n} - 1$ is divisible by 9" for integer $n \geq 0$.

Proof: Base Case: $P(0)$ is true, since $4^{3n} - 1 = 0$, which is divisible by 9.

Inductive Step: Suppose $k \geq 0$ and $P(k)$ is true. So $4^{3k} - 1 = 9m$, for some $m \in \mathbb{Z}$, i.e. $4^{3k} = 9m + 1$

$$\begin{aligned} 4^{3(k+1)} - 1 &= 4^{3k}4^3 - 1 \\ &= 64(9m+1) - 1 && \text{by the inductive hypothesis} \\ &= 9(64m) + 63 \\ &= 9(64m+7) \end{aligned}$$

Since $64m + 7 \in \mathbb{Z}$, we have $P(k+1)$. Thus $P(n)$ is true for every nonnegative integer n, that is, $4^{3n} - 1$ is divisible by 9 for integer $n \geq 0$.

(b) (**3 marks**) Suppose $a+b$ is odd, but a and b are both even. Then $a = 2k$ and $b = 2m$ for some $k, m \in \mathbb{N}$. Thus $a + b = 2k + 2m = 2(k+m)$ is even. A contradiction, so if $a+b$ is odd then a is odd or b is odd.

3. (a) (**6 marks**) $\sim$ is **reflexive** because $3 \mid (x + 2x) = 3x$ for all $x \in S$.

$\sim$ is **symmetric**. Suppose $x \sim y$. Then $3 \mid (x+2y) \iff x + 2y = 3t$ for some $t \in \mathbb{Z}$, and so

$$y + 2x = y + 2(3t - 2y) = 6t - 3y = 3(2t - y).$$

Thus $3 \mid y + 2x$ and $y \sim x$.

$\sim$ is **transitive**. Suppose $(x \sim y) \ \wedge \ (y \sim z)$ for some $x, y, z \in S$. Then $3 \mid (x+2y)$ and $3 \mid (y+2z)$, so that

$$3 \mid (x + 2z) = (x + 2y) + (y + 2z) - 3y.$$

Thus $x \sim z$.

(b) (**4 marks**) $[0] = \{x \in S : 3 \mid x + 2 * 0 = x\} = \{x \in S : 3 \mid x\} = \{-6, 0\}$, $[1] = \{x \in S : 3 \mid x + 2\} = \{-2, 1, 4, 7\}$ and $[5] = \{x \in S : 3 \mid x + 10\} = \{-7, 5\}$.

4. (a) (**6 marks**) $f(a) = f(a') \iff 3a + 5 = 3a' + 5 \iff 3a = 3a' \iff a = a'$, so that f is one-to-one.

Suppose f is onto. Then $f(x) = 0$ for some $x \in \mathbb{Z}$, i.e. $3x + 5 = 0$ and $x = \frac{-5}{3} \notin \mathbb{Z}$. A contradiction. Thus f is not onto.

(b) (**4 marks**) $(g \circ h) \circ (h^{-1} \circ g^{-1}) = g \circ (h \circ h^{-1}) \circ g^{-1} = g \circ 1_{\mathbb{Z}} \circ g^{-1} = g \circ g^{-1} = 1_{\mathbb{Z}}$ and similarly, $(h^{-1} \circ g^{-1}) \circ (g \circ h) = 1_{\mathbb{Z}}$. By definition, $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.