1. $4 \mid 10(3n+8)(7n+27) \iff 10(3n+8)(7n+27) \equiv 0 \pmod 4$. Now $10(3n+8)(7n+27) \equiv 2(3n)(3n+3) \equiv 18n(n+1) \equiv 2n(n+1) \pmod 4$.

If $n = 2m$ is even, then $2n(n+1) = 4m(2m+1) \equiv 0 \pmod 4$.

If $n = 2m+1$ is odd, then $2n(n+1) = 4(2m+1)(m+1) \equiv 0 \pmod 4$.

Thus $10(3n+8)(7n+27) \equiv 0 \pmod 4$ and $4 \mid 10(3n+8)(7n+27)$.

2. $17^{1405} \equiv (-1)^{1405} \equiv -1 \equiv 5 \pmod 6$, so the remainder is 5.

3. Use Euclidean Algorithm we find $\gcd(30,20) = 10 = 30 \cdot 1 + 20 \cdot (-1)$. Thus $350 = 30 \cdot 35 + 20 \cdot (-35)$ and $(35, -35)$ is an integer solution. The general solution of $30x + 20y = 350$ is $x = 35 - \frac{20}{10}t = 35 - 2t$, $y = -35 + \frac{30}{10}t = -35 + 3t$ for $t \in \mathbb{Z}$.

Now $x > 0$ and $y > 0 \iff 35 - 2t > 0$ and $-35 + 3t > 0 \iff \frac{35}{3} < t < \frac{35}{2} \iff t = 12, 13, 14, 15, 16, 17$. It follows that the positive integer solutions are

$$(11,1), (9,4), (7,7), (5,10), (3,13), (1,16).$$

4. Let $d = \gcd(666, 306)$. Then $666x_d + 306y_d = d$ for some $x_d, y_d \in \mathbb{Z}$, and $666x \equiv 180 \pmod{306}$ has a solution $x \in \mathbb{Z} \iff d \mid 180$ and in addition, $x = \frac{180}{d} - \frac{306}{d}t$ for any $t \in \mathbb{Z}$. Thus first we use Euclidean Algorithm to find d and $x_d, y_d \in \mathbb{Z}$:

n	x	y	
666	1	0	r_1
306	0	1	r_2
54	1	-2	$r_3 = r_1 - 2r_2$
36	-5	11	$r_4 = r_2 - 5r_3$
18	6	-13	$r_5 = r_3 - r_4$
0	-17	37	$r_6 = r_4 - 2r_5$

From this we see that $d = 18$, and that $18 = 666 \cdot 6 + 306 \cdot (-13)$. Since $180 = 18 \cdot 10$, it follows that $x = 60 - \frac{306}{18}t = 60 - 17t$ for $t \in \mathbb{Z}$.