1. (a) (i) $-7 = 4(-2) + 1$, and $7 = 4 + 3$, so $f(-7) = 1$ and $f(7) = 3$.

(ii) f is not one-to-one because $f(3) = f(7) = 3$ with $3, 7 \in A$.

(iii) f is not onto because $f(x) \neq -1$ for all $x \in A$.

(b) (i) $g(\{-1, 0, 1\}) = \{f(-1), f(0), f(1)\} = \{3, 0, 1\}$.

(ii) g is not one-to-one because $g(\{3\}) = g(\{7\}) = \{3\}$ with $\{3\}, \{7\} \in \mathcal{P}(A)$.

(iii) g is not onto because $g(X) \neq \{-1\}$ for all $X \in \mathcal{P}(A)$.

(c) (i) $a \in S \iff a \sim 7 \iff f(a) = f(7) = 3 \iff a = 4q + 3$ for some $q \in \mathbb{Z} \iff a - 3 = 4q$ for some $q \in \mathbb{Z}$. Thus

$$S = \{-9, -5, -1, 3, 7\}.$$

(ii) $[0] = \{-8, -4, 0, 8\}$, $[1] = \{-7, -3, 1, 5\}$, $[2] = \{-10, -6, -2, 2, 6\}$ and $[3] = \{-9, -5, -1, 3, 7\}$.

2. (a)

$$\begin{aligned} z \in f(X) \setminus f(Y) &\iff z \in f(X) \wedge z \notin f(Y) \\ &\iff (\exists\, x \in X \ f(x) = z) \wedge (\forall\, y \in Y \ f(y) \neq z) \\ &\implies \exists\, x \in X \setminus Y \ f(x) = z \\ &\iff z \in f(X \setminus Y), \end{aligned}$$

that is, $f(X) \setminus f(Y) \subseteq f(X \setminus Y)$.

(b) Suppose f is one-to-one. It suffices to show that $f(X \setminus Y) \subseteq f(X) \setminus f(Y)$.

Suppose $z \in f(X \setminus Y)$, then $\exists\, x \in X \wedge x \notin Y \ f(x) = z$, so that $z = f(x) \in f(X)$. If $z \in f(Y)$, then $z = f(y)$ for some $y \in Y$ and so $f(x) = f(y) = z$. Since f is one-to-one, it follows that $x = y$ and $x \in Y$, which is impossible. Thus $z \notin f(Y)$ and hence $z \in f(X) \setminus f(Y)$.

Conversely, suppose $f(X) \setminus f(Y) = f(X \setminus Y)$ for all $X, Y \in \mathcal{P}(A)$. Suppose, for a contradiction that f is not one-to-one. Then $f(a) = f(a') = b$ for some distinct $a, a' \in A$. Set $X = \{a\}$ and $Y = \{a'\}$. Then $f(X) = f(Y) = \{b\}$ and $f(X) \setminus f(Y) = \emptyset$. But $X \setminus Y = X$ and $f(X \setminus Y) = f(X) = \{b\}$. A contradiction.

3. Define $f : A \to \mathbb{Z} \setminus \{0\}$ by $f(\frac{1}{2n}) = -n$. Then f is a function.

f **is onto**. For any $b \in \mathbb{Z} \setminus \{0\}$, set $a = \frac{1}{-2b}$. Then $a \in A$ and $f(a) = -(-b) = b$.

f **is order preserving**. Let $a = \frac{1}{2n}$ and $a' = \frac{1}{2n'}$ be two elements of A.

$$\begin{aligned} a \leq a' &\iff \frac{1}{2n} \leq \frac{1}{2n'} \\ &\iff 2n' \leq 2n \\ &\iff n' \leq n \\ &\iff -n \leq -n' \\ &\iff f(a) \leq f(a'). \end{aligned}$$

Thus f is an order isomorphism and so $A \simeq \mathbb{Z} \setminus \{0\}$.