1. For $n \in \mathbb{N}$, let P_n be the statement

$$\text{“}9 \mid (12^n - 3^n)\text{”}.$$

Base case: When $n = 1$, $12^1 - 3^1 = 9 = 9 \cdot 1$. Thus P_1 is true.

Inductive step: For $k \in \mathbb{N}$ suppose P_k is true, that is, $9 \mid (12^k - 3^k)$, which is equivalent to $9c = 12^k - 3^k$ for some $c \in \mathbb{N} \iff 12^k = 9c + 3^k$ for some $c \in \mathbb{N}$.

Want to show that P_{k+1} is true, i.e. $9 \mid (12^{k+1} - 3^{k+1})$. Now

$$\begin{aligned} 12^{k+1} - 3^{k+1} &= 12 \times 12^k - 3^{k+1} \\ &= 12(9c + 3^k) - 3 \times 3^k \\ &= 9 \times 12c + 9 \times 3^k \\ &= 9(12c + 3^k). \end{aligned}$$

Thus P_{k+1} is true and by induction, P_n is true for all $n \in \mathbb{N}$.

2. $x_1 = 1, x_2 = 2, x_3 = x_2 + 2x_2 = 2 + 2 = 4, x_4 = x_3 + 2x_2 = 4 + 4 = 8$ and $x_5 = x_4 + 2x_3 = 8 + 8 = 16$. We conjecture that $x_n = 2^{n-1}$.

For $n \in \mathbb{N}$, let P_n be the statement $x_n = 2^{n-1}$.

Base case: $x_1 = 1 = 2^{1-1}$ and $x_2 = 2 = 2^{2-1}$, so that $x_n = 2^{n-1}$ for $n = 1$ and $n = 2$.

Inductive step: Let $k \in \mathbb{N}$ with $k \geq 2$ and suppose P_i is true for all $1 \leq i \leq k$, that is, $x_i = 2^{i-1}$ for all $1 \leq i \leq k$.

Want to show that P_{k+1} is true, that is, $x_{k+1} = 2^k$. Since $k + 1 \geq 3$,

$$\begin{aligned} x_{k+1} &= x_k + 2x_{k-1} \\ &= 2^{k-1} + 2 \times 2^{k-2} \qquad \text{as} \qquad x_i = 2^{i-1} \\ &= 2 \times 2^{k-1} \\ &= 2^k, \end{aligned}$$

that is, $x_{k+1} = 2^k$. Thus P_{k+1} is true and by complete induction, P_n is true for all $n \in \mathbb{N}$, that is $x_n = 2^{n-1}$.

3. For $n \in \mathbb{N}$ with $n \geq 2$, let P_n be the statement that

$$(A_1 \cap A_2 \cap \ldots \cap A_n)_U^C = (A_1)_U^C \cup (A_2)_U^C \cup \ldots \cup (A_n)_U^C.$$

Base case: When $n = 2$, P_2 is the statement $(A_1 \cap A_2)_U^C = (A_1)_U^C \cup (A_2)_U^C$. Now

$$\begin{aligned} x \in (A_1 \cap A_2)_U^C &\iff x \in U \wedge x \notin (A_1 \cap A_2) \\ &\iff x \in U \wedge (x \notin A_1 \vee x \notin A_2) \\ &\iff (x \in U \wedge x \notin A_1) \vee (x \in U \wedge x \notin A_2) \\ &\iff x \in (A_1)_U^C \vee x \in (A_2)_U^C \\ &\iff x \in (A_1)_U^C \cup (A_2)_U^C, \end{aligned}$$

that is, $(A_1 \cap A_2)_U^C = (A_1)_U^C \cup (A_2)_U^C$ and P_2 is true.

Inductive step: Let $k \in \mathbb{N}$ with $k \geq 2$ and suppose P_k is true, that is,

$$(A_1 \cap A_2 \cap \ldots \cap A_k)_U^C = (A_1)_U^C \cup (A_2)_U^C \cup \ldots \cup (A_k)_U^C.$$

If $T = A_1 \cap A_2 \cap \ldots \cap A_k$, then

$$\begin{aligned}
(A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1})_U^C &= (T \cap A_{k+1})_U^C \\
&= T_U^C \cup (A_{k+1})_U^C \\
&= (A_1 \cap \ldots \cap A_k)_U^C \cup (A_{k+1})_U^C \\
&= (A_1)_U^C \cup \ldots \cup (A_k)_U^C \cup (A_{k+1})_U^C,
\end{aligned}$$

that is, P_{k+1} is true and by induction, P_n is true for all $n \in \mathbb{N}$.

4. For $n \in \mathbb{N}$, let P_n be the statement $3^n > n^2$.

Base case: P_1 is true, because $3^n = 3 > 1^2 = 1$. When $n = 2$, $3^n = 9 > 2^2 = 4$. So P_2 is also true.

Inductive step: Let $k \in \mathbb{N}$ with $k \geq 2$ and suppose P_k is true, that is, $3^k > k^2$. Want to show that $3^{k+1} > (k+1)^2$. Now

$$\begin{aligned}
3^{k+1} &= 3 \times 3^k \\
&> 3 \times k^2 \qquad \text{as} \qquad 3^k > k^2 \\
&= k^2 + k^2 + k^2 \\
&\geq k^2 + 2k + 1 \qquad \text{as} \qquad k \geq 2 \\
&= (k+1)^2,
\end{aligned}$$

that is, $3^{k+1} > (k+1)^2$. Thus P_{k+1} is true and by induction, P_n is true for all $n \in \mathbb{N}$, that is, $3^n > n^2$.