THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2005
Campus: City

MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: This is an OPEN BOOK examination.
Answer ALL **EIGHT** questions. Each question carries 20 marks.

1. For any integer n, let $A(n)$ be the statement:

"If $n = 3q + 2$ or $n = 3q + 1$ for some $q \in \mathbb{Z}$, then $n^2 = 3k + 1$ for some $k \in \mathbb{Z}$."

(a) Write down the negation of $A(n)$. (3 marks)

(b) Write down the contrapositive of $A(n)$. (4 marks)

(c) Write down the converse of $A(n)$. (3 marks)

(d) Use a **direct proof** to show that $(\forall\ n \in \mathbb{Z})\ A(n)$. (5 marks)

(e) Use **proof by contradiction** to show that the converse of $A(n)$ is true for all $n \in \mathbb{Z}$. (5 marks)

2. Let $\sim$ be the relation defined on the set of integers $\mathbb{Z}$ by $x \sim y$ if $8 \mid (3x + 5y)$ for $x, y \in \mathbb{Z}$.

(a) Show that $\sim$ is an equivalence relation. (10 marks)

(b) Find all distinct equivalence classes. (10 marks)

3. (a) Use congruences to show that for any natural number $n \in \mathbb{N}$, the number $21(15n+27)(n+28)$ is divisible by 14. (7 marks)

(b) Suppose a sequence $\{s_n\}_{n=1}^{\infty}$ satisfies $s_1 = 3, s_2 = 18$ and $s_n = 6s_{n-1} - 9s_{n-2}$ for $n \geq 3$. Use complete induction to prove that $s_n = n3^n$ for all $n \in \mathbb{N}$. (13 marks)

4. Let $S = \{2, 4, 5, 6, 8, 10, 15, 18, 20\}$ and let ρ be the relation on S defined by $a \ \rho \ b$ if and only if $a \mid b$. Then (S, ρ) is a poset. [You are not asked to prove this.]

(a) Draw a lattice diagram of (S, ρ). (5 marks)

(b) Find all maximal and all minimal elements of S. (3 marks)

(c) Find a subset of S which has no upper bound and no lower bound. (4 marks)

(d) Find the greatest lower bound for $\{4, 6, 10\}$. (4 marks)

(e) Determine whether or not the subset $\{2, 4, 20\}$ of S is well-ordered. Explain your answer to this part. (4 marks)

5. Let $A = \{x \in \mathbb{R} : x \neq 0\}$ be the set of all non-zero real numbers, and let $S = \{x \in \mathbb{Q} : x \neq 0\}$ be the set of all non-zero rational numbers and $T = \mathbb{R} \setminus \mathbb{Q}$. For any $x, y \in \mathbb{R}$ define $x * y$ by

$$x * y = 3xy,$$

where xy is the ordinary multiplication of x and y in $\mathbb{R}$.

(a) Show that $(A, *)$ is an abelian group. (10 marks)

(b) Show that $(S, *)$ is a subgroup of $(A, *)$. (5 marks)

(c) Determine with reason whether or not $(T, *)$ is a subgroup of $(A, *)$. (5 marks)

6. (a) Find all integer solutions of the Diophantine equation

$$946x + 374y = 44$$

with $0 < x < 22$. (13 marks)

(b) Find all integers $x \in \mathbb{Z}$ such that

$$189x \equiv 28 \pmod{56}.$$

(7 marks)

7. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers and define the sequence $\{c_n\}_{n=1}^{\infty}$ by

$$c_n = \begin{cases} a_n & \text{if } b_n \leq a_n, \\ b_n & \text{if } a_n < b_n. \end{cases}$$

Suppose that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = a$$

for some real number $a \in \mathbb{R}$. Prove from the first principles that $\lim_{n \to \infty} c_n = a$. (20 marks)

8. Let f be a function from $\mathbb{R}$ to itself defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0, \\ x^3 & \text{if } 0 < x \leq 1, \\ x + 2 & \text{if } 1 < x. \end{cases}$$

(a) Prove from the first principles that $f(x)$ is continuous at 0. (10 marks)

(b) Prove from the first principles that $f(x)$ is *not* continuous at 1. (10 marks)