1. (a) **Reflexive**: $\forall X \in S\ x \in X \implies (\exists x \in X\ x|x)$. Thus $X \sim X$.

Not antisymmetric: Take $X = \{1, 2\}$ and $Y = \{2\}$. Then $X \sim Y$ and $Y \sim X$, but $X \neq Y$.

Not symmetric: Take $X = \{2\}$ and $Y = \{2, 3\}$. Then $X \sim Y$, but $Y \not\sim X$ since $3 \in Y$ but $3 \not| 2$.

Transitive: $X \sim Y \wedge Y \sim Z \iff (\forall x \in X\ \exists y \in Y\ x|y) \wedge (\forall y \in Y\ \exists z \in Z\ y|z)$. If $x \in X$ and $x|y$ for some $y \in Y$, then $y|z$ for some $z \in Z$, so $x|z$. Thus $\forall x \in X\ \exists z \in Z\ x|z$, and $X \sim Z$.

(b) **Not reflexive**: Take $x = 1 \in B$. Then $1 + 1 = 2 \neq 0$, so $x \not\sim x$.

Antisymmetric: If $x, y \in B$ with $x \sim y$ and $y \sim x$, then $x + y = y + x = 0$, so that $x = y = 0$.

Symmetric: $x \sim y \iff x + y = 0 \iff y + x = 0 \iff y \sim x$.

Transitive: $(x \sim y \wedge y \sim z) \iff (x + y = 0 \wedge y + z = 0) \iff (x = y = z = 0) \implies x + z = 0$.

2. (a) For all $x \in S$, $(x, x) \in \rho$, so ρ is reflexive.

For all $x, y \in S$, $(x, y) \in \rho \implies (y, x) \in \rho$, so ρ is symmetric.

For all $x, y, z \in S$, $((x, y) \in \rho \wedge (y, z) \in \rho) \implies (x, z) \in \rho$, so ρ is transitive.

$[a] = \{a, b, c\}$, $[d] = \{d, e\}$ and $[f] = \{f\}$.

(b) $S_i \neq \emptyset$ for each i, $S_i \cap S_j = \emptyset$ for $i \neq j$ and $S = S_1 \cup S_2 \cup S_3$. So $\{S_1, S_2, S_3\}$ is a partition.

$$\rho = \{(a, a)(b, b), (c, c), (d, d), (e, e), (f, f), (b, d), (d, b), (b, f), (f, b), (d, f), (f, d)(c, e), (e, c)\}.$$

3. (a) **Reflexive**: $\forall x \in A\ x + 3x = 4x$ is even so $x \sim x$.

Symmetric: $x \sim y \iff x + 3y$ is even $\iff x + 3y = 2a$ for some $a \in \mathbb{Z}$. Thus $y + 3x = y + 3(2a - 3y) = 6a - 8y = 2(3a - 4y)$ is even, so $y \sim x$.

Transitive: $x \sim y \wedge y \sim z \iff (x + 3y = 2a) \wedge (y + 3z = 2b)$ for some $a, b \in \mathbb{Z}$. Thus $(x + 3y) + (y + 3z) = 2(a + b)$, so $x + 3z = 2(a + b - 2y)$ is even and $x \sim z$.

(b) $x \in [0] \iff x + 3 \times 0$ is even iff x is even. $x \in [1] \iff x + 3$ is even iff x is odd. Thus $\mathbb{Z} = [0] \cup [1]$ and so $[0], [1]$ are all the distinct equivalence classes.