1. (**5 marks**) For $n \in \mathbb{N}$, $6 | 9(n+12)(11n+25) \iff 9(n+12)(11n+25) \equiv 0 \pmod{6}$. Now

$$\begin{aligned} 9(n+12)(11n+25) &\equiv 3n(5n+1) \\ &\equiv -3n(n-1) \pmod{6}. \end{aligned}$$

If n is even, then $6|3n$. If n is odd, then $6|3(n-1)$. So in any case $6|-3n(n-1)$ and hence $6|9(n+12)(11n+25)$.

2. (a) (**5 marks**) If $f(x) = 0$ or $g(x) = 0$, then $f(x)g(x) = 0$ and

$$\deg(f(x)g(x)) = -\infty = \deg(f(x)) + \deg(g(x)).$$

(Note $-\infty + c = -\infty$ for any integer c or $c = -\infty$.)
Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ such that $\deg(f(x)) = n \neq -\infty$ and $\deg(g(x)) = m \neq -\infty$, so that $a_n \neq 0$ and $b_m \neq 0$. Then

$$f(x)g(x) = c_0 + c_1x + \cdots + c_{n+m}x^{n+m},$$

where $c_{n+m} = a_nb_m$. Since $a_n \neq 0$ and $b_m \neq 0$, it follows that $c_{n+m} \neq 0$, and so $\deg(f(x)g(x)) = m + n = \deg(f(x)) + \deg(g(x))$.

(b) (**5 marks**) Since $a(x) \mid b(x)$ and $b(x) \mid a(x)$, it follows that $a(x) = 0 \iff b(x) = 0$.
If $a(x) = 0 = b(x)$, then $a(x) = cb(x)$ for some non-zero $c \in \mathbb{R}$.
Suppose $a(x) \neq 0$, so that $b(x) \neq 0$. Since $b(x) \mid a(x)$, it follows that $b(x)u(x) = a(x)$ for some $0 \neq u(x) \in \mathbb{R}[x]$, so that by (a) above,

$$\deg(b(x)) + \deg(u(x)) = \deg(a(x))$$

and in particular, $\deg(b(x)) \leq \deg(a(x))$. Similarly, since $a(x) \mid b(x)$, it follows that $\deg(a(x)) \leq \deg(b(x))$, and hence $\deg(a(x)) = \deg(b(x))$. In particular, $\deg(u(x)) = 0$ and $u(x) = c$ is a non-zero number.

(c) (**5 marks**) Let $n = 4$, and $c(x) = \bar{2}x + \bar{1} = d(x) \in \mathbb{Z}_4[x]$. Then

$$c(x)d(x) = (\bar{2}x + \bar{1})^2 = \bar{4}x^2 + \bar{4}x + \bar{1} = \bar{1}$$

and so $\deg(c(x)d(x)) = 0 \neq 2 = \deg(c(x)) + \deg(d(x))$.

3. (a) (**6 marks**) Using long division in $\mathbb{Z}_5[x]$ we have

$$x^4 + 2x^3 + 4x^2 + 2x + 3 = (4x^3 + 2x^2 + 4x + 2)(4x + 1) + (x^2 + 1),$$

so that $q(x) = 4x + 1$ and $r(x) = x^2 + 1$.

(b) (**6 marks**) Use Euclidean Algorithm:

$f(x)$	$u(x)$	$v(x)$	
$x^4 + 2x^3 + 4x^2 + 2x + 3$	1	0	r_1
$4x^3 + 2x^2 + 4x + 2$	0	1	r_2
$x^2 + 1$	1	$-(4x+1)$	$r_3 = r_1 - (4x+1)r_2$
0	*	*	$r_4 = r_2 - (4x+2)r_3$

It follows that x^2+1 is a $\gcd(f(x), g(x))$ and

$$x^2+1 = (x^4+2x^3+4x^2+2x+3) + (4x^3+2x^2+4x+2)(-4x-1),$$

so that $u(x) = 1$ and $v(x) = -4x-1 = x+4$.

4. (**8 marks**) Let $A = \begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} y & y \\ 0 & 0 \end{pmatrix}$ be elements of K.

Then $AB = \begin{pmatrix} xy & xy \\ 0 & 0 \end{pmatrix}$ and since $x \neq 0$ and $y \neq 0$ we have $xy \neq 0$, thus $AB \in K$.

Hence, K is closed under matrix multiplication.

Associativity is satisfied for multiplication of *all* matrices, not only for the ones in K.

The identity element (which element E **in** $\boldsymbol{K}$ satisfies $EX = XE = X$ for all $\in K$?) is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and it is a member of K.

The inverse of $A = \begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix}$ **in** $\boldsymbol{K}$ is $\begin{pmatrix} 1/x & 1/x \\ 0 & 0 \end{pmatrix}$ (note that $x \neq 0$, so $1/x$ makes sense).

Thus K is a group.