DEPARTMENT OF MATHEMATICS

MATHS 255 Assignment 6 Solution 16 September, 2003

1. Determine whether or not the following subsets of the poset (**R,** ≤) are order isomorphic. ["Determine" means prove your answer]

   (a). $\mathbf{R}$, $\mathbf{R}^+$ (i.e. the positive real numbers).

   (b) $\mathbf{R}$, $(0,1)$ (i.e the open interval).

   (c) $S = \left\{\frac{1}{n}: n \in \mathbf{Z}\setminus\{0\}\right\}$, $T = \mathbf{Z}\setminus\{0\}$.

Soln:

(a) Yes, let $f(x) = e^x$. Then $f: \mathbf{R} \to \mathbf{R}^+$ is a strictly order-preserving bijection.

(b) Yes, let $f(x) = \frac{\tan^{-1} x}{\pi/2}$ for example. Then $f: \mathbf{R} \to (0,1)$ is a strictly order-preserving bijection.

(c) No. We can prove this by finding some poset property that *S* has but not *T*. For example, *S* has a smallest element (= –1) but *T* has no smallest element.

2. Let $S = \mathbf{Q}\setminus\{1\}$.

   (a) Show that * defined as follows is a binary operation on *S*: $a * b = a + b - ab$. [You must show that if $a,b \in S$ then $a * b \in S$.]

Soln: The sum and product of rational numbers is rational, so we only need to show that if $a \neq 1$ and $b \neq 1$ then $a*b \neq 1$. So suppose $a*b = 1$. $a+b-ab=1$ implies $a(1-b)=1-b$, hence $(a-1)(1-b)=0$, so $a=1$ or $b=1$.

(b) Show that * is commutative and associative.

Soln: $a*b = a+b-ab = b+a-ba = b*a$. Hence, * is commutative.

$a*(b*c)=a+(b*c)-a(b*c) = a+(b+c-bc)-a(b+c-bc) = a+b+c-ab-bc-ac+abc.$

$(a*b)*c = a*b+c-(a*b)c = a+b-ab+c-(a+b-ab)c = a+b+c-ab-ac-bc+abc =a*(b*c).$

(c) Find an identity element *e* under the operation *.

Soln: An identity element *e* would have to satisfy at least the condition that $e*e = e$. In other words $e = e+e-ee$. This implies $e = ee$ so that $e = 0$ or 1. So if an identity element exists, it must be 0. Now $a*0 = a+0-a0 = a$, so sure enough, $e = 0$ is an identity element.

(d) Prove that your answer to (c) is unique (i.e. that if *e, f* are both identity elements under the operation *, then $e = f$.

Soln: $e = e*f = f$. (The first equation follows since *f* is an identity, and the second follows because *e* is an identity.)

(e) Show that every element of *S* has an "inverse", i.e. $\forall x \in S\ \exists y \in S,\ x * y = e$ (where *e* is the unique identity element from (c) and (d) above).

Soln: $x*y = e$ implies $x+y-xy = 0$ hence $x(y-1)=y$ and $x=y/(y-1)$. Since $y \in S$, we have $y \neq 1$, so that $y/(y-1)$ is a defined rational number. Moreover, $y/(y-1) \neq 1$ since it is impossible that $y = y-1$. Since $y*(y/(y-1)) = 0 = e$, we see that any element *y* has an inverse under *.

3. Prove that the product of any four consecutive positive integers is divisible by 12.

Soln: Any four consecutive integers contain at least one which is divisible by 3 and exactly two which are divisible by 2. Hence their product is divisible by both 3 and 4, so that the product must be divisible by 12 (since gcd(3,4) = 1). Alternatively, this can be proved by induction: For n > 0 let P(n) be the statement: n(n+1)(n+2)(n+3) is divisible by 4. P(I) is true because 1*2*3*4 = 24 = 12*2. If k > 1 and P(k) is true, then k(k+1)(k+2)(k+3) = 12m for some integer m. One then uses this to show that (k+1)(k+2)(k+3)(k+4) is divisible by 12. Etc. It is kind of messy and not the best way.

4. Use the Euclidean algorithm to find $d = \gcd(m,n)$ and to find integers *u,v* such that $d = mu + nv$, and use prime factorization to find lcm(*m,n*) if $m = 4635$ and $n = 17061$.

Soln:

4635	1	0
17061	0	1
3156	-3	1
1479	4	-1
198	-11	3
93	81	-22
12	-173	47
9	1292	-351
3	-1465	398

0 Last non-zero remainder is 3.

Hence gcd(4635, 17061) = 3 and 3 = 4635*(-1465) + 17061*(398).

$4635 = 3^2 * 5 * 103$, and $17061 = 3 * 11^2 * 47$. Hence, $lcm(4635,17061) = 3^2 * 5 * 11^2 * 47 * 103$.