1. [*8 marks*] From the results in class (or by experimentation or some other method), we claim that $\sum_{i=1}^{n} 3(4i^2 + 2i - 1) = n(4n+1)(n+2) = 4n^3 + 9n^2 + 2n$.

Proof: We proceed to prove the sum by induction.

Base case: When $n = 1$, $3(4n^2 + 2n - 1) = 15 = n(4n+1)(n+2)$ as required.

Inductive step: Suppose $\sum_{i=1}^{k} 3(4i^2 + 2i - 1) = k(4k+1)(k+2)$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} 3(4i^2 + 2i - 1) &= \sum_{i=1}^{k} 3(4i^2 + 2i - 1) + 3(4(k+1)^2 + 2(k+1) - 1) \\
&= 4k^3 + 9k^2 + 2k + 12(k^2 + 2k + 1) + 6k + 6 - 3 \\
&= 4k^3 + 21k^2 + 32k + 15 \\
&= (k+1)(4k^2 + 17k + 15) \\
&= (k+1)(4k+5)(k+3) \\
&= (k+1)(4(k+1)+1)((k+1)+2), \text{ as required.}
\end{aligned}$$

So, by mathematical induction, $\sum_{i=1}^{n} 3(4i^2 + 2i - 1) = n(4n+1)(n+2)$.

2. [*7 marks*]

Proof: We proceed to prove that the inequality holds when $n \geq 3$. Let $|A| = n$. We note that $|\mathcal{P}(A \times A)| = 2^{n^2}$, while $|\mathcal{P}(A) \times \mathcal{P}(A)| = 2^{2n}$.

Base case: When $n = 3$, we have $2^9 \geq 2^6$, as required. The base case cannot be $n = 2$, since $2^4 \not> 2^4$.

Inductive step: Suppose $2^{k^2} \geq 2^{2k}$, for some $k \geq 3$. Then $k^2 > 2k$. Now

$$\begin{aligned}
2(k+1) &= 2k + 2 \\
&< 2k + k \\
&< k^2 + k \\
&< k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

So, by mathematical induction, $|\mathcal{P}(A \times A)| > |\mathcal{P}(A) \times \mathcal{P}(A)|$ when $|A| \geq 3$.

3. [*7 marks*]

Proof:

Base case: When $n = 1$, we have $7^{n+2} - 3^n = 343 - 3 = 340 = 4 \times 85$, so $4|(7^{n+2} - 3^n)$.

Inductive step: Assume $7^{k+2} - 3^k = 4m$, for some integer m. Then

$$\begin{aligned} 7^{k+3} - 3^{k+1} &= 7 \times 7^{k+2} - 3 \times 3^k \\ &= 4 \times 7^{k+2} + 3(7^{k+2} - 3^k) \\ &= 4 \times 7^{k+2} + 3(4m) \\ &= 4(7^{k+2} + 3m), \end{aligned}$$

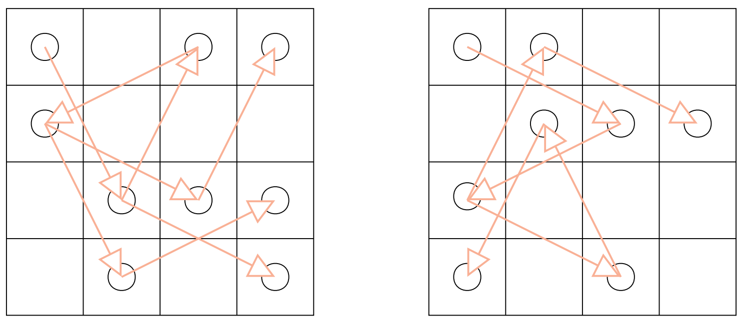
so it has 4 as a divisor.

So, by mathematical induction, $7^{n+2} - 3^n$ has 4 has a divisor, for any natural number n.

4. [*10 marks*]

Proof:

Base case: Consider the 4×4 chessboards shown. Together, they give a path of knight moves from the top-left square to any other square. Since paths are obviously reversible, these give a path of knight moves from any square to any other.



Inductive Step: Suppose that there is a path of knight moves from any square to any square on an $n \times n$ chessboard.

Suppose A and B are two squares of an $(n+1) \times (n+1)$ chessboard. We note that the A lies in an $n \times n$ chessboard with the square $C = (2, 2)$. Similarly B lies in an $n \times n$ chessboard with the square C, although not necessarily the same $n \times n$ chessboard.

So there is a path of knight moves from A to C, and a path of knight moves from B to C, by the inductive hypothesis. Combining these paths gives a path from A to B, as required.

So, by mathematical induction, there is a path of knight moves from any square to any square on any $n \times n$ chessboard with $n \neq 4$.

5. [*8 marks*]

(a) $\lhd$ is reflexive, since for any $x \in \mathbb{R}$, we have $|x - x| = 0 < 1$. So $x \lhd x$.

(b) $\lhd$ is symmetric. Suppose $x, y \in \mathbb{R}$ and $|x - y| < 1$. Then $y - x = -(x - y)$, and so $|y - x| = |x - y| < 1$. So $x \lhd y \implies y \lhd x$.

(c) $\lhd$ is not antisymmetric, since $0.1 \lhd 0$ and $0 \lhd 0.1$, but $0 \neq 0.1$.

(d) $\lhd$ is not transitive, since $0 \lhd 0.9$ and $0.9 \lhd 1.8$, but $0 \not\lhd 1.8$.

TOTAL MARKS: 40