1. [*8 marks*]

Existence: Suppose $x \in \mathbb{R}$. Then since $f(x)g_k(x) = k$, and $f(x) \neq 0$, we have $g_k(x) = \frac{k}{f(x)}$. This construction of $g_k(x)$ can be applied at any $x \in \mathbb{R}$, since $f(x) \neq 0$.

Uniqueness: Suppose there are two functions $g(x)$ and $h(x)$ for which $f(x)g(x) = f(x)h(x) = k$, for every x. Then clearly, for any x, we have $g(x) = \frac{k}{f(x)} = h(x)$, so $g(x) = h(x)$ for every x, and so the functions are equal. Hence, $g_k(x)$ us unique.

2. [*4 marks*]

When $f(\alpha) = 0$, there are two cases to consider.

If $k = 0$, then at $x = \alpha$, we have $f(\alpha)g_0(\alpha) = 0$, so $0g_0(\alpha) = 0$; any value of $g_0(\alpha)$ will work, and $g_0(x)$ is not unique.

For $k \neq 0$, at $x = \alpha$ we have $f(\alpha)g_k(\alpha) = k$, so $0g_k(\alpha) = k \neq 0$. No value for $g_k(\alpha)$ can, when multiplied by 0, give $k \neq 0$. So the function $g_k(x)$ does not exist.

3. [*13 marks*]

(a) Direct Proof: Let $h(x) = f(x) \cdot g(x)$. Now we consider $h(-x)$.

$$\begin{aligned} h(-x) &= f(-x) \cdot g(-x) \\ &= f(x) \cdot g(x) && \text{by the even-ness of } f(x) \text{ and } g(x) \\ &= h(x) \end{aligned}$$

so $h(x)$ is an even function.

The converse is not true. For example, $f(x) = g(x) = x$ is not an even function, but $h(x) = f(x)g(x) = x^2$ is an even function.

(b) Direct Proof: Let $h(x) = f(x) + g(x)$. Now consider $h(-x)$.

$$\begin{aligned} h(-x) &= f(-x) + g(-x) \\ &= -f(x) + -g(x) && \text{by the oddness of } f(x) \text{ and } g(x) \\ &= -(f(x) + g(x)) \\ &= -h(x) \end{aligned}$$

so $h(x)$ is an odd function.

The converse is not true. For example, if $f(x) = 2x - 1$ and $g(x) = 3x + 1$, then $h(x) = f(x) + g(x) = x$ is odd, but neither of $f(x)$ and $g(x)$ is odd.

(c) Direct Proof: If $f(x)$ is even and odd, then $f(-x) = f(x) = -f(x)$. So, for any $x \in \mathbb{R}$, we have $f(-x) = -f(x)$. This means that $f(-x) = 0$, for any x. So $f(x) = 0$ is the unique function which is both even and odd.

4. [*10 marks*]

(a) (i) TRUE
(ii) FALSE
(iii) FALSE
(iv) TRUE
(v) FALSE
(vi) TRUE
(vii) TRUE
(viii) TRUE

(b) There are different possible answers, perhaps the most compact being:

$$B = \{C \subseteq A : |C| = 2\}$$

5. [*6 marks*]

Commutative: need to show $A \oplus B = B \oplus A$.

If $x \in A \oplus B$, then either $x \in A$ and $x \notin B$, or $x \notin A$ and $x \in B$. In either case, $x \in B \oplus A$. So $A \oplus B \subseteq B \oplus A$. The converse is exactly the same, so $A \oplus B = B \oplus A$.

Associative: need to show that $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

Suppose $x \in (A \oplus B) \oplus C$.

Then either (a) $x \in A \oplus B$ and $x \notin C$, or (b) $x \in C$ and $x \notin A \oplus B$.

In case (a), either $x \in A$ or $x \in B$, but not both. If $x \in A$, then $x \notin B$. Since $x \notin B$ and $x \notin C$, we have $x \notin B \oplus C$, so $x \in A \oplus (B \oplus C)$. Otherwise $x \in B$, and we have $x \notin A$ and $x \in B \oplus C$, so $x \in A \oplus (B \oplus C)$.

In case (b), either $x \in A$ and $x \in B$, or $x \notin A$ and $x \notin B$. If it is the first of these, then $x \notin B \oplus C$, and so $x \in A \oplus (B \oplus C)$. In the second case, $x \in B \oplus C$ and so $x \in A \oplus (B \oplus C)$ also.

In any case, $x \in A \oplus (B \oplus C)$, so $(A \oplus B) \oplus C \subseteq A \oplus (B \oplus C)$.

By symmetry, the proof of $A \oplus (B \oplus C) \subseteq (A \oplus B) \oplus C$ is the same (we can make use of commutativity), so they are equal, as required.

[A 'proof' showing these sets as Venn diagrams is not sufficient.]

TOTAL MARKS: 40