NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

PLEASE SHOW ALL WORKING.

1. (**15 marks**) Let a and b be natural numbers. Suppose $a = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ and $b = p_1^{f_1} p_2^{f_2} \cdots p_\ell^{f_\ell}$, where $p_1, \cdots, p_\ell$ are distinct primes and $e_i, f_i \geq 0$.

(a) (**5 marks**) Show that $a|b$ if and only if $e_i \leq f_i$ for all i.

(b) (**5 marks**) Show that $\gcd(a, b) = p_1^{m_1} p_2^{m_2} \cdots p_\ell^{m_\ell}$, where $m_i = \min\{e_i, f_i\}$.

(c) (**5 marks**) Show that $\operatorname{lcm}(a, b) = p_1^{g_1} p_2^{g_2} \cdots p_\ell^{g_\ell}$, where $g_i = \max\{e_i, f_i\}$. Check that $\operatorname{lcm}(a, b) \gcd(a, b) = ab$.

2. (**15 marks**)

(a) Find all solutions to the following Diophantine equations:

(i) (**5 marks**) $2598x + 604y = 14$.

(ii) (**5 marks**) $2598x + 604y = 12$.

(b) (**5 marks**) Find all solutions to the Diophantine equation $2598x + 604y = 12$ with $10 \leq x \leq 200$

3. (**10 marks**)

(a) (**5 marks**) Find all integers $x \in \mathbb{Z}$ such that

$$3x^2 - x - 4 \equiv 0 \pmod{5}.$$

(b) (**5 marks**) Find all integers $x \in \mathbb{Z}$ such that

$$35x \equiv 14 \pmod{42}.$$