

Tuesday: Functions as relations, one to one and onto functions

What is a function? [5.1]

Informally, a function from A to B is a rule which assigns to each element a of A a unique element $f(a)$ of B . Officially, we have

Definition. A function f from A to B is a subset of $A \times B$ such that

- for each $a \in A$ there is a $b \in B$ with $(a, b) \in f$.
- if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

We write $f : A \rightarrow B$ to show that f is a function from A to B . If $a \in A$, we write $f(a)$ for the unique $b \in B$ such that $(a, b) \in f$.

Thus if f is a function, we have $f(a) = b \iff (a, b) \in f$. We will use this equivalence later.

Definition. If $f : A \rightarrow B$, the A is the domain of f and B is the codomain of f . We write $\text{Dom}(f)$ for A and $\text{Codom}(f)$ for B . We also define the range of f , $\text{Ran}(f)$, by

$$\text{Ran}(f) = \{ b \in B : (\exists a \in A)(f(a) = b) \}.$$

Note that $\text{Ran}(f) \subseteq \text{Codom}(f)$, but there are examples where the two sets are not the same.

Definition (Equality of functions). Two functions $f : A \rightarrow B$ and $g : A' \rightarrow B'$ are equal iff they are the same set of ordered pairs, in other words iff $A = A'$ and $f(a) = g(a)$ for all $a \in A$. [Notice that the textbook also requires that $B = B'$: most authors would consider that f and g are equal even if the codomains differ, as long as the domain and values are the same.]

One-to-one and onto [5.1]

Definition. A function $f : A \rightarrow B$ is one-to-one if for each $b \in B$ there is at most one $a \in A$ with $f(a) = b$. It is onto if for each $b \in B$ there is at least one $a \in A$ with $f(a) = b$. It is a one-to-one correspondence or bijection if it is both one-to-one and onto.

Notice that “ f is one-to-one” is asserting uniqueness, while “ f is onto” is asserting existence. This gives us the idea of how to prove that functions are one-to-one and how to prove they are onto.

Example 1. Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2x + 1$ is one-to-one and onto.

Example 2. Show that the function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(n) = 2n + 1$ is one-to-one but not onto.

For functions from \mathbb{R} to \mathbb{R} , we can use the “horizontal line test” to see if a function is one-to-one and/or onto. The horizontal line $y = b$ crosses the graph of $y = f(x)$ at precisely the points where $f(x) = b$. So f is one-to-one if no horizontal line crosses the graph more than once, and onto if every horizontal line crosses the graph at least once.

Example 3. Sketch graphs of the following functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and determine whether each is one-to-one and/or onto.

1. $f(x) = x^3 + x$.
2. $f(x) = x^3 - x$.
3. $f(x) = e^x$.
4. $f(x) = x^2$.

Notice that in the definition of “onto”, we need to know what the codomain is. So the function $f = \{(x, e^x) : x \in \mathbb{R}\}$ is not onto when thought of as a function from \mathbb{R} to \mathbb{R} , but it is onto when thought of as a function from \mathbb{R} to $(0, \infty)$.

Proposition 4. Let $f : A \rightarrow B$ be a function. Then f is an onto function from A to $\mathcal{Ran}(f)$. If f is one-to-one, then f is a bijection from A to $\mathcal{Ran}(f)$.

Wednesday: Composition of functions, Inverses

Composition of functions [5.2]

Definition. Let $f : A \rightarrow B$ and $g : B \rightarrow C$. We define a new function $g \circ f : A \rightarrow C$ by declaring that $(g \circ f)(a) = g(f(a))$. We call $g \circ f$ “ g composed with f ”.

Example 5. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2 + 2$ and $g(x) = 3x$. Find $g \circ f$ and $f \circ g$.

The above example shows that $f \circ g$ and $g \circ f$ need not be equal. Of course, if A and C are not the same, they will not even be defined: if $g(b) \notin A$ then trying to figure out what $f(g(b))$ is gives a type error. [For example, “my mother’s telephone number” makes sense but “my telephone number’s mother” does not.

Composition of functions interacts with the notions of one-to-one and onto: it does preserve these properties, and in some cases if the composition has the property then so must the original functions.

Theorem 6. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions.

1. If f and g are both one-to-one then $g \circ f$ is one-to-one.
2. If f and g are both onto then $g \circ f$ is onto.
3. If $g \circ f$ is one-to-one then f is one-to-one.
4. If $g \circ f$ is onto then g is onto.

However there are examples of f and g with $g \circ f$ both one-to-one and onto but g not one-to-one and f not onto.

Although \circ is not commutative, it is associative.

Theorem 7. Let $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$ are functions then $(h \circ (g \circ f)) = ((h \circ g) \circ f)$.

Proof. See pp 110 and 111 of the textbook, Problem 5.2.6. □

Definition. Let A be a set. The identity function on A , 1_A , is the function $1_A : A \rightarrow A$ given by $f(a) = a$ for all $a \in A$. In other words, $1_A = \{(a, a) : a \in A\}$.

Proposition 8. Let $f : A \rightarrow B$ be a function. Then $1_B \circ f = f = f \circ 1_A$.

Thursday: Inverses, images and preimages

Inverses [5.2]

Definition. Let $f : A \rightarrow B$ be a function. An inverse of f is a function $g : B \rightarrow A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

Example 9. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = 2x + 1$ and $g(x) = \frac{1}{2}x - \frac{1}{2}$. Then g is an inverse of f .

Example 10. Let $f : \mathbb{R} \rightarrow [0, \infty)$ and $g : [0, \infty) \rightarrow \mathbb{R}$ be given by $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then g is **not** an inverse of f . Although we have $f(g(x)) = (\sqrt{x})^2 = x$, so $f \circ g = 1_{[0, \infty)}$, when we consider $g(f(x)) = \sqrt{x^2}$, we always get the positive square root, so for example $g(f(-2)) = \sqrt{(-2)^2} = \sqrt{4} = 2$.

Lemma 11. Let $f : A \rightarrow B$. Suppose g is an inverse of f . Then $g = \{(f(a), a) : a \in A\}$.

Proof. We must prove two inclusions: $g \subseteq \{(f(a), a) : a \in A\}$ and $\{(f(a), a) : a \in A\} \subseteq g$.

Let $p \in g$. Then $p = (b, g(b))$ for some $b \in B$. Now $f(g(b)) = b$, so we have $p = (f(g(b)), g(b))$, i.e. $p = (f(a), a)$ where $a = g(b)$. So $p \in \{(f(a), a) : a \in A\}$ as required.

Conversely, let $a \in A$: we will show that $(f(a), a) \in g$. Now, we know that $g(f(a)) = a$. Since $g(y) = z \iff (y, z) \in g$, this means that $(f(a), a) \in g$, as required. \square

Lemma 12. Let $f : A \rightarrow B$ be a function. If f has an inverse, it is unique.

Proof. Use Proposition 8 and Theorem 7. \square

If f has an inverse, we write it as f^{-1} .

Theorem 13. Let $f : A \rightarrow B$ be a function. Then f has an inverse if and only if f is a bijection.

Proof. Suppose first that f has an inverse. We must show that f is one-to-one and onto.

One-to-one: Let $x, y \in A$ with $f(x) = f(y)$. Then $f^{-1}(f(x)) = f^{-1}(f(y))$, i.e. $x = y$, as required.

Onto: Let $b \in B$. Then $f(f^{-1}(b)) = b$, i.e. there is at least one $a \in A$ (namely $a = f^{-1}(b)$) such that $f(a) = b$.

Conversely, suppose that f is a bijection. We must find a candidate function g such that g is an inverse of f . From the earlier lemma, there is only one possible choice: we must have $g = \{(f(a), a) : a \in A\}$. So we must check that this is indeed an inverse of f : we must show it is a function, that $f \circ g = 1_B$ and that $g \circ f = 1_A$.

Function: Let $b \in B$. On the one hand, since f is onto, there is at least one $a \in A$ with $f(a) = b$, so there is at least one $a \in A$ with $(f(a), a) = (b, a)$, so there is at least one $a \in A$ with $(b, a) \in g$. On the other hand, since f is one-to-one, there is at most one $a \in A$ with $f(a) = b$, so there is at most one $a \in A$ with $(b, a) \in g$. So g is indeed a function.

$f \circ g = 1_B$: Let $b \in B$. Put $a = g(b)$. So $(b, a) \in g$, so $(b, a) = (f(c), c)$ for some $c \in A$. We must have $c = a$, so $b = f(a)$, i.e. $b = f(g(b))$, as required.

$g \circ f = 1_A$: Let $a \in A$. Then $(f(a), a) \in g$, so $g(f(a)) = a$, as required.

□

Images and preimages [5.3]

Definition. Let $f : A \rightarrow B$ be a function. For $S \subseteq B$ we define the inverse image or preimage of S under f to be

$$f^{-1}(S) = \{ a \in A : f(a) \in S \}.$$

Notice that $f^{-1}(S)$ is a **subset** of A , not an element of A . It is defined whether or not f^{-1} exists as a function. Note that there is a slight ambiguity here: if f happens to be a bijection then $f^{-1}(b)$ is an element of A for $b \in B$ and $f^{-1}(S)$ is a subset of A for $S \subseteq B$. Since the elements of B are not (usually) subsets of B , this ambiguity should never cause a problem.

Example 14. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$. Find the following sets:

1. $f^{-1}(\{4\})$.
2. $f^{-1}([-2, 9])$.
3. $f^{-1}((1, 4])$.
4. $f^{-1}(\{-9\})$.

To prove facts about preimages, we use the equivalence that

$$x \in f^{-1}(S) \iff f(x) \in S.$$

Example 15. Let $f : A \rightarrow B$ be a function, $S, T \subseteq B$. Then

1. $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
2. $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.
3. $f^{-1}(S_B^c) = f^{-1}(S)_A^c$.

In the same way as the preimage, we define the image of a subset of A :

Definition. Let $f : A \rightarrow B$ be a function and $S \subseteq A$. We define the image of S under f to be

$$f(S) = \{ f(a) : a \in S \}.$$

Example 16. Let $f : A \rightarrow B$ be a function, $S, T \subseteq B$. Then

1. $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
2. $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

Notice that in the second of these, we only get \subseteq rather than $=$: we can find examples with $f(S \cap T) \subset f(S) \cap f(T)$.

Friday: Orders and functions

Definition. Let (A, \preceq_A) and (B, \preceq_B) be sets. A function $f : A \rightarrow B$ is order-preserving if for all $x, y \in A$,

$$x \preceq_A y \implies f(x) \preceq_B f(y).$$

It is strictly order preserving if for all $x, y \in A$,

$$x \preceq_A y \iff f(x) \preceq_B f(y).$$

For example, a constant function (in other words a function f such that there is some $b \in B$ with $f(x) = b$ for all $x \in A$) is order preserving, but is not strictly order preserving unless A is empty or has only one element.

Proposition 17. If (A, \preceq_A) and (B, \preceq_B) are posets and $f : A \rightarrow B$ is strictly order preserving then f is one-to-one.

Definition. Let (A, \preceq_A) and (B, \preceq_B) be posets. An order-isomorphism from A to B is a bijection $f : A \rightarrow B$ such that f and f^{-1} are both order-preserving. If there exists such an isomorphism, we say that A and B are order-isomorphic.

Example 18. Let $A = \{n \in \mathbb{N} : n \mid 30\}$ and $B = \mathcal{P}(\{2, 3, 5\})$. Define $f : A \rightarrow B$ by $f(n) = \{m \in \{2, 3, 5\} \mid m \mid n\}$. Then f is an order-isomorphism from (A, \mid) to (B, \subseteq) .

Example 19. Show that (\mathbb{Z}, \leq) is order-isomorphic to (E, \leq) , where E is the set of even integers.

Theorem 20. Let (A, \preceq_A) and (B, \preceq_B) be posets and $f : A \rightarrow B$ a bijection. Then f is an order-isomorphism iff f is strictly order preserving.

Theorem 21. Let A and B be posets and $f : A \rightarrow B$ an order-isomorphism. Then $x \in A$ is maximal in A iff $f(x)$ is maximal in B .

Example 22. Let $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$, and $B = A \cup \{1\}$, with the usual order they get as subsets of \mathbb{R} . Then \mathbb{N} is order isomorphic to A but \mathbb{N} is not order-isomorphic to B .