

Tuesday: Orderings

Properties of relations [4.1]

Example 1 (4.1.10). Consider the following relations. Which, if any, of the four properties reflexive, symmetric, antisymmetric and transitive, do these relations have?

1. $A = \mathbb{N}$, $x \sim y$ if $x + y$ is even.
2. $A = \mathbb{N}$, $x \sim y$ if $x + y$ is odd.
3. $A = \mathcal{P}(\mathbb{N})$, $x \sim y$ if $x \subseteq y$.
4. $A = \mathbb{R}$, $x \sim y$ if $x = 2y$.
5. $A = \mathbb{R}$, $x \sim y$ if $x - y$ is rational.

Example 2. Recall that for $m, n \in \mathbb{Z}$, $m \mid n$ if there is some $a \in \mathbb{Z}$ with $n = ma$. Show that \mid is reflexive and transitive and not symmetric. Show that \mid is not antisymmetric on \mathbb{Z} but is antisymmetric on \mathbb{N} .

Partial and total orderings [4.2]

Definition. A relation ρ on a set A is a partial order (or partial ordering) if ρ is reflexive, antisymmetric and transitive.

If ρ is a partial order on A , we say that A is a partially ordered set under ρ , or that the pair (A, ρ) is a partially ordered set. We often abbreviate “partially ordered set” to poset.

Example 3. From Example 2 we see that \mid is a partial order on \mathbb{N} .

Example 4. Let A be a set, and let $S = \mathcal{P}(A)$. Then S is a poset under the relation \subseteq .

We have the convention of using the symbol \leq for a typical partial order. Sometimes we will use the symbol \preceq instead, particularly when the usual order on the real numbers is being used in the definition of our order.

Example 5. Let $A = \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and define a relation \preceq on A by declaring that

$$(u, v) \preceq (x, y) \iff u \leq x \wedge v \leq y.$$

Then \preceq is a partial order.

Proof. We must show \preceq is reflexive, antisymmetric and transitive.

Reflexive: Let $(x, y) \in A$. Then $x \leq x$ and $y \leq y$ so $(x, y) \preceq (x, y)$.

Antisymmetric: Let $(x, y), (u, v) \in A$ with $(x, y) \preceq (u, v)$ and $(u, v) \preceq (x, y)$. Then $x \leq u$ and $y \leq v$, and $u \leq x$ and $v \leq y$. Combining these we have $x \leq u \leq x$, so $x = u$, and $y \leq v \leq y$, so $y = v$. Hence $(x, y) = (u, v)$, as required.

Transitive: Let $(x, y), (u, v), (z, w) \in A$ with $(x, y) \preceq (u, v)$ and $(u, v) \preceq (z, w)$. Then $x \leq u$ and $y \leq v$, and $u \leq z$ and $v \leq w$. Combining these we have $x \leq u \leq z$, so $x \leq z$, and $y \leq v \leq w$, so $y \leq w$. Hence $(x, y) \preceq (z, w)$, as required.

□

The name “partial” order is supposed to allow the possibility that not all the elements are ranked in the order: for example, in the previous example we have pairs $(1, 5)$ and $(3, 3)$ which are *incomparable*: we have both $(1, 5) \not\preceq (3, 3)$ and $(3, 3) \not\preceq (1, 5)$ (the first is because $5 \not\leq 3$ and the second because $3 \not\leq 1$).

Definition. A total order or total ordering on a set A is a partial order \preceq on A with the additional property that for any $a, b \in A$ we have $a \preceq b$ or $b \preceq a$.

Wednesday: Lattice diagrams, maximal elements and greatest elements

Lattice diagrams [4.2]

A good way to represent a (reasonably small) poset is with a *lattice diagram*. To understand these, first we need some notation. If \preceq is a partial order on A , we define a new relation \prec on A by

$$x \prec y \iff x \preceq y \wedge x \neq y.$$

Similarly, if \leq is a partial order on A we define a new relation $<$ by declaring that $x < y \iff x \leq y$ and $x \neq y$.

Definition. Let \preceq be a partial order on A , and let $a, b \in A$. Then b covers a if $a \prec b$ and there does not exist any $c \in A$ with $a \prec c \prec b$.

Example 6. In the poset \mathbb{Z} , with its usual ordering, $n + 1$ covers n for each n .

To draw a lattice diagram for a poset A , we put a blob for each element of A , and an edge going up from a to b if b covers a . We can then see that for $a, b \in A$, $a \prec b$ if there is a path going upwards from a to b .

Example 7. Let $A = \{n \in \mathbb{N} : n \mid 30\} = \{1, 2, 3, 5, 6, 10, 15, 30\}$. Draw a lattice diagram for the poset $(A, |)$.

Example 8. Draw a lattice diagram for the poset $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$.

Maximal and minimal, greatest and least elements [4.2]

Definition. Let (A, \preceq) be a poset, and let $a \in A$. Then a is maximal if there is no $b \in A$ with $a < b$, and minimal if there is no $b \in A$ with $b < a$. Further, a is a greatest element if $b \preceq a$ for all $b \in A$ and a least element if $a \preceq b$ for all $b \in A$.

Example 9. Draw a lattice diagram for (A, mid) , where $A = \{1, 2, 3, \dots, 10\}$. Identify any maximal and minimal elements and any greatest and least elements.

Theorem 10. *If a poset A has a greatest element, then it is unique. Likewise, if A has a least element then it is unique.*

Note that we can find examples where A has no greatest or least element, and we can find examples where there are more one maximal elements or minimal elements.

Every **finite** partial order has at least one maximal element and one minimal element. However, the poset (\mathbb{Z}, \leq) has no maximal or minimal elements.

Thursday: Least upper bounds and partitions

Upper bounds and lower bounds [4.2]

Definition. *Let A be a poset and $S \subseteq A$. Then u is an upper bound for S if $(\forall a \in S)(a \leq u)$, and l is a lower bound for S if $(\forall a \in S)(l \leq a)$.*

Example 11. *Let $A = \{1, 2, \dots, 10\}$, partially ordered by $|$. Find any upper and lower bounds of the sets $\{2, 3\}$, $\{2, 5\}$ and $\{3, 5\}$.*

Definition. *Let A be a poset and let $S \subseteq A$. An element $s \in A$ is a least upper bound or supremum for S if*

- s is an upper bound for S , and
- if u is also an upper bound for S then $s \leq u$.

If S has a supremum, then the supremum is unique. However, there are some partial orders with subsets which have no supremum, either because S has no upper bounds at all or because there is no least element in the set of upper bounds of S .

Definition. *Let A be a poset. We say that A has the least upper bound property if every nonempty subset of A with at least one upper bound has a least upper bound.*

Example 12. *The poset (\mathbb{R}, \leq) has the least upper bound property: this is one of the axioms for the real numbers we will see in Chapter 8.*

Example 13. *The poset $(\mathbb{N}, |)$ has the least upper bound property: we will prove this in Chapter 6.*

Partitions [4.3]

Definition. *Let S be a set. Then Ω is a partition of S if*

- $\Omega \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$; and
- if $A, B \in \Omega$ with $A \neq B$ then $A \cap B = \emptyset$; and
- $S = \bigcup_{A \in \Omega} A$.

In other words, Ω is a pairwise disjoint family of nonempty subsets of S such that every element of S is in one of the sets in Ω .

Example 14. Let $O = \{2n + 1 : n \in \mathbb{Z}\}$, $E = \{2n : n \in \mathbb{Z}\}$. Then $\Omega = \{O, E\}$ is a partition of \mathbb{Z} .

Example 15. The family $\{[n, n + 1) : n \in \mathbb{Z}\}$ is a partition of \mathbb{R} .

Example 16. Find all possible partitions of $\{1, 2, 3\}$.

Friday: Equivalence relations

The set of relatives

Let \sim be a relation on a set A . For $a \in A$, define the set of relatives of a to be

$$T_a = \{x \in A : a \sim x\}.$$

Notice that \sim is reflexive iff $a \in T_a$ for all $a \in A$. So if \sim is reflexive then $\Omega = \{T_a : a \in A\}$ will be a family of nonempty subsets of A whose union is A .

Definition. An equivalence relation on a set A is a relation on A which is reflexive, symmetric and transitive.

Example 17. Define \sim on \mathbb{Z} by declaring that for $m, n \in \mathbb{Z}$ we have

$$m \sim n \iff 5 \mid m - n.$$

Then \sim is an equivalence relation.

Theorem 18. If \sim is an equivalence relation on A and $a, b \in A$ then the following are equivalent:

1. $a \sim b$.
2. $T_a \cap T_b \neq \emptyset$.
3. $T_a = T_b$.

Theorem 19. If \sim is an equivalence relation on A then $\Omega = \{T_a : a \in A\}$ is a partition of A .

Conversely, if Ω is a partition of A then the relation \sim defined by declaring that

$$x \sim y \iff (\exists A \in \Omega)(x \in A \wedge y \in A)$$

is an equivalence relation and $\Omega = \{T_a : a \in A\}$.