Tuesday: Existence proofs and counterexamples

Existence proofs [1.9]

To prove something of the form "there is an x such that $A(x)$", we do two steps:

- produce a suitable value of x (like pulling a rabbit from a hat)
- show that that particular value of x does what is claimed.

Example 1. *Show that there is some $x \in \mathbb{R}$ such that $x^2 + 12x - 85 = 0$.*

Proof. Let $x = 5$. Then $x^2 + 12x - 85 = 5^2 + 12 \cdot 5 - 85 = 25 + 60 - 85 = 0$, as required. □

Uniqueness proofs [1.10]

To prove that there is at most one x with the property $A(x)$, we suppose that we have two objects x and y with $A(x)$ and $A(y)$, and deduce that $x = y$.

Lemma 2. *If $x, y \in \mathbb{R}$ with $x^2 + xy + y^2 = 0$ then $x = y = 0$.*

Proof. Exercise. Hint: $x^2 + xy + y^2 = \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2$. □

Example 3. *Cube roots are unique, in other words if r is a real number then there is at most one $x \in \mathbb{R}$ with $x^3 = r$.*

Proof. Suppose that $x, y \in \mathbb{R}$ with $x^3 = r$ and $y^3 = r$. Then $x^3 - y^3 = r - r = 0$, and $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$. Now, if $a, b \in \mathbb{R}$ with $ab = 0$ then $a = 0$ or $b = 0$, so $x - y = 0$ or $x^2 + xy + y^2 = 0$. Now if $x - y = 0$ then $x = y$, and if $x^2 + xy + y^2 = 0$ then $x = y = 0$, by the Lemma. So □

Examples and counterexamples [1.11]

Remember when we want to prove an implication $A(x) \implies B(x)$, we are really proving the statement $(\forall x)(A(x) \implies B(x))$. To show that the implication is not a theorem, we are proving $\sim(\forall x)(A(x) \implies B(x))$, i.e. $(\exists x)(A(x) \land \sim B(x))$. So what we have to do is give an existence proof. Again, we find an object x and then demonstrate that it has the properties $A(x)$ and $\sim B(x)$. Such an object is called a *counterexample* to the implication $A(x) \implies B(x)$.

Example: Exercise 1.11.1

Wednesday: Sets, subsets, set equality

Sets and Set notation [2.1]

A *set* is a collection of objects. We write $x \in A$ if the object x is in the set, otherwise $x \notin A$. We can specify a set in three ways:

- enumerate the elements, e.g. $X = \{1, 2, 3\}$, $Y = \{1, 3, 5, \ldots, 17\}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- use set builder notation, e.g. $X = \{\, x \in \mathbb{N} : 1 \leq x \leq 3 \,\}$, $Y = \{\, n \in \mathbb{N} : n \text{ is odd and } 1 \leq n \leq 17 \,\}$, $\mathbb{N} = \{\, x : x \text{ is a natural number} \,\}$.
- Use an indexing set, e.g. $Y = \{\, 2n - 1 : n \in \{1, 2, \ldots, 9\} \,\}$.

Some sets are so important they have their own names, e.g. $\mathbb{N}$, Z, $\mathbb{R}$, $\mathbb{Q}$ and intervals such as $[a, b]$, $[a, b)$, (a, b) and $(-\infty, b)$. One other set with a name: the *empty set* $\emptyset$.

Subsets [2.2]

A *subset* of a set A is a set S with the property that every element of S is also an element of A. We write $S \subseteq A$.

Examples: $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Q} \subseteq \mathbb{R}$. For any set X, $\emptyset \subseteq X$ and $X \subseteq X$.

Important: do not mix up $x \in A$ and $x \subseteq A$.

Notice that $S \subseteq A$ is an implication: "if $x \in S$ then $x \in A$".

Exercise 2.2.4.

A *proper subset* of a set A is a set S with $S \subseteq A$ and $S \neq A$. We will sometimes write $S \subset A$ in this case. Warning: some books use $S \subset A$ to mean S is a subset of A, not necessarily a proper subset of S.

To say that two sets A and B are equal is to say that they have exactly the same elements, i.e. that $A \subseteq B$ and $B \subseteq A$. So to prove that two sets are equal, we have to prove two implications.

Example: to show that $\{\, x \in \mathbb{R} : x^2 + 12x - 85 = 0 \,\} = \{5, -17\}$ we have to prove two implications:

- if $x \in \mathbb{R}$ with $x^2 + 12x - 85 = 0$ then $x = 5$ or $x = -17$; and
- if $x = 5$ or $x = -17$ then $x \in \mathbb{R}$ with $x^2 + 12x - 85 = 0$.

Thursday: Set operations

Complement, intersection and union [2.3]

Given a set U (which we call a *universal set*) and a set $S \subseteq U$, we define the *complement* of S in U to be $S^{\mathcal{C}}_U$. If U is fixed and understood, we may simply write $S^{\mathcal{C}}$ and refer to the *complement* of S.

Example (exercise 2.3.2 and 2.3.3). Put $S = [-5, 2]$, $U = [-5, 5]$. Find $S_U^{\mathcal{C}}$ and $S_{\mathbb{R}}^{\mathcal{C}}$.

Definition: if A and B are sets then the *intersection* of A and B is $A \cap B = \{\, x : x \in A \wedge x \in B \,\}$ and the *union* of A and B is $\{\, x : x \in A \vee x \in B \,\}$.

Example (exercise 2.3.5): let $A = \{a, b, c, d, e, f, g\}$, $B = \{a, e, i, o, u\}$. Find $A \cap B$ and $A \cup B$.

We may use *Venn diagrams* to illustrate these.

Set identities [2.4]

Recall that to show that two sets are equal we have to prove two implications.

Example 4. *Let A and B be sets. Show that $A \cap (A \cup B) = A$.*

Proof. Let $x \in A \cap (A \cup B)$. Then ... so $x \in A$.

Conversely, let $y \in A$. Then ... so $y \in A \cap (A \cup B)$. □

Example (Theorem 2.4.2): for any sets A, B and C we have $A \cup (B \cap C) = A \cup B) \cap (A \cup C)$.

Set operations with indexing sets

Suppose we have a set Λ, and for each $\alpha \in \Lambda$ we have a set U_α. Then we may form the union of all these sets and (provided $\Lambda \neq \emptyset$) the intersection of all these sets. We define the union to be

$$\bigcup_{\alpha \in \Lambda} U_\alpha = \{\, x : x \in U_\alpha \text{ for at least one } \alpha \in \Lambda \,\}$$

and the intersection to be

$$\bigcap_{\alpha \in \Lambda} U_\alpha = \{\, x : x \in U_\alpha \text{ for every } \alpha \in \Lambda \,\}.$$

Example: for each $n \in N$ let $I_n = [0, \frac{1}{n}]$. Find $\bigcap_{n \in \mathbb{N}} I_n$ and $\bigcup_{n \in \mathbb{N}} I_n$.

Example: find $\bigcap_{n \in \mathbb{Z}} [n, n+1]$ and $\bigcup_{n \in \mathbb{N}} [n, n+1]$.

Friday: The power set

Exercise: list all the subsets of $\{1, 2, 3\}$.

The collection of all subsets of a set A is called the *power set* of A, written $\mathcal{P}(A)$. So we have $S \in \mathcal{P}(A)$ if and only if $S \subseteq A$.

Example 5 (Theorem 2.5.4). *Show that if A and B are sets then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Example 6 (Theorem 2.5.5). *Let A and B be sets. Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.*

Example 7. *Let A and B be sets. Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Find an example of sets A and B such that $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$*