1. (a) "If $n+1$ is prime then n is even".

(b) "If $n+1$ is composite then n is odd".

(c) "n is even and $n+1$ is composite".

2. (a) Suppose a is even or b is even. If a is even, then $a = 2k$ for some $k \in \mathbb{Z}$, so $a^2b = (2k)^b = 2(2k^2b)$, and $2k^2b \in \mathbb{Z}$, so a^2b is even. If b is even then $b = 2j$ for some $j \in \mathbb{Z}$, so $a^2b = 2(a^2j)$ and $a^2j \in \mathbb{Z}$, so a^b is even in this case also. So either way, a^2b is even.

(b) Suppose, for a contradiction, that a^2b is even but neither a nor b is even. Then a and b are both odd, so $a = 2k+1$ and $b = 2j+1$ for some $k, j \in \mathbb{Z}$. But then

$$\begin{aligned} a^2b = (2k+1)^2(2j+1) &= (4k^2+4k+1)(2j+1) \\ &= 8k^2j + 8kj + 2j + 4k^2 + 4k + 1 = 2(4k^2j + 4kj + j + 2k^2 + 2k) + 1 \end{aligned}$$

and $4k^2j + 4kj + j + 2k^2 + 2k \in \mathbb{Z}$, so a^2b is odd, contradicting our assumption that a^2b is even.

3. (a) Suppose f is one-to-one. Let $x, y \in \mathbb{R}$ with $g(x) = g(y)$. Then $f(x+2) = f(y+2)$, so (since f is one-to-one) $x+2 = y+2$, so $x = y$. Hence g is one-to-one.

Conversely, suppose g is one-to-one. Let $x, y \in \mathbb{R}$ with $f(x) = f(y)$. Put $x' = x-2$, $y' = y-2$. Then $x = x'+2$ and $y = y'+2$, so $f(x'+2) = f(y'+2)$, i.e. $g(x') = g(y')$. Thus $x' = y'$, i.e. $x-2 = y-2$, so $x = y$. Hence f is one-to-one.

(b) Suppose that g is onto, and let $b \in \mathbb{R}$. Then there is some $a \in \mathbb{R}$ with $g(a) = b$. So $f(a+2) = b$. Put $a' = a+2$. Then $f(a') = b$, as required.

4. For $n \in \mathbb{N}$, let P_n be the statement "$n^2 + 3n + 1$ is odd".

Base case: for $n = 1$ we have $n^2 + 3n + 1 = 1 + 3 + 1 = 5$ which is odd, so P_1 is true.

Inductive step: let $n \in \mathbb{N}$, and suppose P_n is true. Then n^2+3n+1 is odd, so $n^2+3n+1 = 2k+1$ for some k. But then

$$\begin{aligned} (n+1)^2 + 3(n+1) + 1 &= n^2 + 2n + 1 + 3n + 3 + 1 \\ &= (n^2 + 3n + 1) + 2n + 4 \\ &= 2k + 1 + 2n + 4 \\ &= 2(k + n + 2) + 1, \end{aligned}$$

and $k + n + 2 \in \mathbb{Z}$, so $(n+1)^2 + 3(n+1) + 1$ is odd, so P_{n+1} is true.

Hence, by induction, P_n is true for all n.

5. Let $x \in X$. Then $x \mid a$ and $x \mid b$, so there exist $p, q \in \mathbb{Z}$ with $a = px$ and $b = qx$. But then $a + b = px + qx = (p + q)x$, and $p + q \in \mathbb{Z}$, so $x \mid a + b$. Since we already know that $x \mid a$, we have $x \mid a \wedge x \mid b$, so $x \in Y$. Hence $X \subseteq Y$.

Conversely, let $y \in Y$. Then $y \mid a$ and $y \mid a + b$, so there exist $r, s \in \mathbb{Z}$ with $a = ry$ and $a + b = sy$. Then $b = (a + b) - a = sy - ry = (s - r)y$, and $s - r \in \mathbb{Z}$, so $y \mid b$. Since we already knew that $y \mid a$ we have $y \in X$. Hence $Y \subseteq X$.

Combining these we have $X = Y$, as required.