1. We may compute the Cayley Table either by working out the effects of doing one symmetry followed by the other, or by observing that each of the operations is its own inverse, which lets us fill in the entries on the diagonal of the Cayley Table and then using the "once per row and once per column" rule to fill in the remaining entries. Either way, we get the Cayley Table

$\circ$	R_0	R_{180}	V	H
R_0	R_0	R_{180}	V	H
R_{180}	R_{180}	R_0	H	V
V	V	H	R_0	R_{180}
H	H	V	R_{180}	R_0

2. (a) We have $\alpha = (1\ 2\ 3\ 6)(4\ 5)$.

(b) We have $\beta\alpha\beta^{-1} = (1\ 2\ 3)(1\ 2\ 3\ 6)(4\ 5)(3\ 2\ 1) = (1\ 6\ 2\ 3)(4\ 5)$.

3. We certainly have $g\circ f : G \to K$. Put $h = g\circ f$: we must check that for all $x, y \in G$, $h(xy) = h(x)h(y)$. So, let $x, y \in G$. Then

$$\begin{aligned} h(xy) &= g(f(xy)) \\ &= g(f(x)f(y)) && \text{(since } f \text{ is a homomorphism)} \\ &= g(f(x))g(f(y)) && \text{(since } g \text{ is a homomorphism)} \\ &= h(x)h(y), \end{aligned}$$

as required.

4. Suppose first that f is one-to-one. [We will show that $f^{-1}(\{e_H\}) = \{e_G\}$.] Let $x \in f^{-1}(\{e_H\})$. Then $f(x) \in \{e_H\}$, so $f(x) = e_H$. We also have $f(e_G) = e_H$, since f is a homomorphism, so $f(x) = f(e_G)$, and f is one-to-one so $x = e_G$, so $x \in \{e_G\}$. Thus $f^{-1}(\{e_H\}) \subseteq \{e_G\}$. Conversely, let $y \in \{e_G\}$. Then $y = e_G$, so $f(y) = f(e_G) = e_H$, so $f(y) \in \{e_H\}$, so $y \in f^{-1}(\{e_H\})$. Thus $\{e_G\} \subseteq f^{-1}(\{e_H\})$, so $f^{-1}(\{e_H\}) = \{e_G\}$.

Conversely, suppose that $f^{-1}(\{e_H\}) = \{e_G\}$. [We will show that f is one-to-one.] Let $x, y \in G$ with $f(x) = f(y)$. Then

$$\begin{aligned} f(x)f(y)^{-1} &= e_H \\ f(x)f(y^{-1}) &= e_H \\ f(xy^{-1}) &= e_H, \end{aligned}$$

so $f(xy^{-1}) \in \{e_H\}$, so $xy^{-1} \in f^{-1}(\{e_H\}) = \{e_G\}$, so $xy^{-1} = e_G$, so $x = y$. Thus f is one-to-one, as required.

5. (a) Put $H = \{e, x\}$.

Suppose first that H is a subgroup of G. Then, since $x \in H$ we must have $xx \in H$, i.e. $x^2 \in \{e, x\}$, so $x^2 = e$ or $x^2 = x$. We cannot have $x^2 = x$, since $x \neq e$, so $x^2 = e$ as required.

Conversely, suppose that $x^2 = e$. We must check that $e \in H$, that if $g, h \in H$ then $gh \in H$, and that if $g \in H$ then $g^{-1} \in H$. Well, we certainly have $e \in H$. We consider all possible values of g and h: $ee = e \in H$, $ex = x \in H$, $xe = x \in H$ and $x^2 = e \in H$. Finally, we have $e^{-1} = e \in H$ and $x^{-1} = x \in H$. Thus H is a subgroup, as required.

(b) Suppose first that G has an element x with $x \neq e$ and $x^2 = e$. Then $H = \{e, x\}$ is a subgroup of G, and by Lagrange's Theorem that means that $|H|$ divides $|G|$, i.e. 2 divides $|G|$, so $|G|$ is even.

Conversely, suppose that $|G|$ is even. Put $A = \{\, g \in G : g^2 \neq e \,\}$.

Claim: if $g \in A$ then $g \neq g^{-1}$.

For: suppose, for a contradiction, that $g \in A$ but $g^{-1} = g$. Then $g^{-1}g = gg$, i.e. $e = g^2$, so $g \notin A$, contradicting our assumption.

Claim: if $g \in A$ then $g^{-1} \in A$.

For: suppose, for a contradiction, that $g \in A$ but $g^{-1} \notin A$. Then $(g^{-1})^2 = e$, so $(g^{-1}) {-} 1 = g^{-1}$, i.e. $g = g^{-1}$, contradicting the previous claim.

From this, we can see that A can be partitioned into a collection of disjoint pairs $\{g, g^{-1}\}$. If there are k such pairs, then $|A| = 2k$, so $|A|$ is even. Since $A \subseteq G$ we have $|G \setminus A| = |G| - |A|$, and $|G|$ and $|A|$ are both even, so $|G \setminus A|$ is even. So $|G \setminus A|$ is either 0 or at least 2. But $|G \setminus A| \neq 0$, since $e \in G \setminus A$. So $|G \setminus A| \geq 2$, so there is at least one element in $G \setminus A$ besides e. In other words, there is some $x \in G$ with $x \neq e$ and $x^2 = e$, as required.