1. (a) We first use the extended Euclidean Algorithm to find $\gcd(35, 12)$:

n	x	y	
35	1	0	r_1
12	0	1	r_2
11	1	-2	$r_3 = r_1 - 2r_2$
1	-1	3	$r_4 = r_2 - r_3$
0	12	-35	$r_5 = r_3 - 11r_4$

From this we see that $\gcd(35, 12) = 1$, and that $1 = 35 \cdot (-1) + 12 \cdot 3$, so $16 = 35 \cdot (-16) + 12 \cdot 48$. So the general solution is $x = -16 - 12t$, $y = 48 + 35t$ for $t \in \mathbb{Z}$.

(b) Again, we start by using the extended Euclidean Algorithm to find $\gcd(30, 12)$:

n	x	y	
30	1	0	r_1
12	0	1	r_2
6	1	-2	$r_3 = r_1 - 2r_2$
0	-2	5	$r_4 = r_2 - 2r_3$

From this we see that $\gcd(30, 24) = 6$. Since $6 \nmid 15$, the equation $30x + 12y = 15$ has no integer solutions.

(c) From our working in part (b) we know that $\gcd(30, 12) = 6$, and $6 = 30 \cdot 1 + 12 \cdot (-2)$, and $18 = 3 \cdot 6$, so $18 = 30 \cdot 3 + 12 \cdot (-6)$. Thus the general solution is $x = 3 - \frac{12}{6}t = 3 - 2t$, $y = -6 + \frac{30}{6}t = -6 + 5t$ for $t \in \mathbb{Z}$.

2. First, suppose that $a \equiv b \pmod{n}$. Then $n \mid a - b$, so there exists $s \in \mathbb{Z}$ with $a - b = sn$. Now, $b = q_b n + r_n(b)$ for some $q_b \in \mathbb{Z}$, so we have $a = b + sn = q_b n + r_n(b) + sn = (q_b + s)n + r_n(b)$. By the uniqueness part of the division algorithm, since $q_b + s, r_n(b) \in \mathbb{Z}$ and $0 \leq r_n(b) < n$, we must have $r_n(a) = r_n(b)$.

Conversely, suppose that $r_n(a) = r_n(b)$. There exist $q_a, q_b \in \mathbb{Z}$ with $a = q_a n + r_n(a)$, $b = q_b n + r_n(b)$. Then

$$a - b = (q_a n + r_n(a)) - (q_b n + r_n(b)) = (q_a - q_b)n + (r_n(a) - r_n(b)) = (q_a - q_b)n.$$

Thus, since $q_a - q_b \in \mathbb{Z}$, $n \mid a - b$, so $a \equiv b \pmod{n}$.

3. Solving $\overline{33} = \overline{47} \cdot_{250} \overline{x}$ is equivalent to solving $33 = 47x + 250y$, so we first solve this equation. We use the extended Euclidean Algorithm to find $\gcd(250, 47)$:

n	y	x	
250	1	0	r_1
47	0	1	r_2
15	1	-5	$r_3 = r_1 - 5r_2$
2	-3	16	$r_4 = r_2 - 3r_3$
1	22	-117	$r_5 = r_3 - 7r_4$
0	-47	250	$r_6 = r_4 - 2r_5$

From this we see that $\gcd(250, 47) = 1$ and that $1 = 47\cdot(-117)+25\cdot22$, so $33 = 47\cdot(-3861)+250\cdot726$. The general solution of $33 = 47x + 250y$ is $x = -3861 + 250t$, $y = 726 - 47t$ for $t \in \mathbb{Z}$, so the general solution of $\overline{33} = \overline{47} \cdot_{250} \overline{x}$ is $\overline{x} = \overline{-3861 + 250t}$ for $t \in \mathbb{Z}$. Adding multiples of 250 does not change equivalence classes: we choose a suitable value of t to give a value between 0 and 249, namely $\overline{x} = \overline{-3861 + 4000} = \overline{139}$.

4. Suppose that b and n are not relatively prime. Put $d = \gcd(b, n)$, so $d > 1$. Notice that for any $x \in \mathbb{Z}$ we have

$$\begin{aligned} \overline{a} = \overline{b} \cdot_n \overline{x} \quad &\iff \quad \overline{a} = \overline{bx} \\ &\iff \quad a \equiv bx \pmod{n} \\ &\iff \quad a = bx + ny \text{ for some } y \in \mathbb{Z} \end{aligned}$$

If $d \nmid a$ then the equation $a = bx + ny$ cannot have any solutions, so $\overline{a} = \overline{b} \cdot_n \overline{x}$ does not have any solutions. If $d \mid a$, say $a = dq$, then the general solution of $a = bx + ny$ is $x = qx_d - \frac{n}{d}t$, $y = qy_d + \frac{b}{d}t$ for $t \in \mathbb{Z}$, where $d = bx_d + ny_d$. Choosing $t = 0$ and $t = 1$ gives us two solutions, $x_0 = qx_d$ and $x_1 = qx_d - \frac{n}{d}$. Now $d > 1$, so $1 \leq \frac{n}{d} < n$. But then $x_0 - x_1 = \frac{n}{d} \not\equiv 0 \pmod{n}$. Thus $\overline{x_0} \neq \overline{x_1}$, so in this case the solution to $\overline{a} = \overline{b} \cdot_n \overline{x}$ is not unique.

Thus, if b and n are not relatively prime then $\overline{a} = \overline{b} \cdot_n \overline{x}$ has either no solutions or more than one solution: either way, it does not have a unique solution. Hence, by contraposition, if $\overline{a} = \overline{b} \cdot_n \overline{x}$ has a unique solution then b and n are relatively prime.