1. (a) Suppose e and f are both identity elements, so that for all $a \in A$ we have $a * e = e * a = a$ and $a * f = f * a = a$. Applying $e * a = a$ with $a = f$ gives $e * f = f$. Applying $a * f = a$ with $a = e$ gives $e * f = e$. So $e = e * f = f$, so $e = f$. [Notice that we can't go from $e * a = a$ and $f * a = a$ to $e * a = f * a$ to $e = f$: the last step is not allowed, because we have no reason to think there is a cancellation law for $*$.]

(b) Suppose that y and z are both inverses, so we have $x * y = y * x = e$ and $x * z = z * x = e$. Then

$$z = z * e = z * (x * y) = (z * x) * y = e * y = y,$$

so $z = y$ as required.

(c) Define the functions $f, g : \mathbb{Z} \to \mathbb{Z}$ by declaring that

$$f(n) = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \qquad g(n) = 2n.$$

Then, for any $n \in \mathbb{N}$, $g(n) = 2n$ is even, so $f(g(n)) = f(2n) = \frac{1}{2}(2n) = n$. Thus $f \circ g = 1_{\mathbb{Z}}$. However, 1 is not even so $f(1) = 0$, so $g(f(1)) = g(0) = 0$, so $g \circ f \neq 1_{\mathbb{Z}}$.

2. We have

$$\begin{aligned} ((a \cdot b) + ((-a) \cdot b)) &= ((b \cdot a) + (b \cdot (-a))) && (\cdot \text{ is commutative}) \\ &= b(a + (-a)) && (\text{multiplication distributes over addition}) \\ &= b \cdot 0 && (a + (-a) = 0) \\ &= 0 && (\text{result from lectures}) \end{aligned}$$

Thus we have $((a \cdot b) + ((-a) \cdot b)) = 0$, so $(-a) \cdot b = -(a \cdot b)$.

3. (a) We have $z = pb = p(qg) = q(pg) = qa$.

(b) Since $z = qa$ we have $a \mid z$. Since $z = pb$ we have $b \mid z$. So z is a common multiple of a and b, so $l \mid z$, in other words there is some $t \in \mathbb{Z}$ with $z = lt$. Note that z and l are both positive, so t is also positive, i.e. $t \in \mathbb{N}$ as required.

(c) We have $zg = (pb)g = (pg)b = ab$.

(d) Substituting $z = lt$ into the above we have $ab = (lt)g = t(lg)$, so $lg \mid ab$.

(e) We have $g = ax + by$, so $gl = axl + byl$. Substituting $l = bn$ in one place and $l = am$ in another we get $gl = axbn + byam$.

(f) From the above we have $gl = axbn + byan = ab(xn + ym)$, so $ab \mid gl$.

(g) From (d) and (f) we have $gl \mid ab$ and $ab \mid gl$, so (since both ab and gl are positive) $ab = gl$.

4. (a) The algorithm gives us

n	x	y	
128	1	0	r_1
60	0	1	r_2
8	1	-2	$r_3 = r_1 - 2r_2$
4	-7	15	$r_4 = r_2 - 7r_3$
0	15	-32	$r_5 = r_3 - 2r_2$

From which we see that $\gcd(128, 60) = 4$ and $4 = 128 \cdot (-7) + 60 \cdot 15$.

(b) The algorithm gives us

n	x	y	
42	1	0	r_1
16	0	1	r_2
10	1	-2	$r_3 = r_1 - 2r_2$
6	-1	3	$r_4 = r_2 - r_3$
4	2	-5	$r_5 = r_3 - r_4$
2	-3	8	$r_6 = r_4 - r_5$
0	8	-21	$r_7 = r_5 - 2r_6$

from which we see that $\gcd(42, 16) = 2$ and $2 = 42 \cdot (-3) + 16 \cdot 8$.

(c) The algorithm gives us

n	x	y	
1230	1	0	r_1
153	0	1	r_2
6	1	-8	$r_3 = r_1 - 8r_2$
3	-25	201	$r_4 = r_2 - 25r_3$
0	51	-410	$r_5 = r_3 - 2r_4$

from which we see that $\gcd(1230, 153) = 3$ and $3 = 1230 \cdot (-25) + 153 \cdot 201$.