1. Suppose first that F is onto. We must show that f is onto, so let $b \in B$. Then $\{b\} \subseteq B$, so there is some $S \in \mathcal{P}(A)$ with $F(S) = \{b\}$. Since $b \in \{b\}$, $b \in F(S) = \{\, f(a) : a \in S \,\}$, so there is some $a \in S$ with $f(a) = b$, as required.

Conversely, suppose that f is onto. Let $Y \in \mathcal{P}(B)$. We must find some $X \in \mathcal{P}(A)$ with $F(X) = Y$. So let $X = f^{-1}(Y) = \{\, a \in A : f(a) \in Y \,\}$. We will show that $F(X) = Y$. So, let $x \in F(X) = \{\, f(a) : a \in X \,\}$. Then $x = f(a)$ for some $a \in X$. Since $a \in X$, we have $f(a) \in Y$, i.e. $x \in Y$. Thus $F(X) \subseteq Y$. Conversely, let $y \in Y$. Since f is onto, there is some $a \in A$ with $f(a) = y$. Then $f(a) \in Y$ since $y \in Y$, so $a \in X$. Thus $f(a) \in F(X)$, i.e. $y \in F(X)$. Hence $Y \subseteq F(X)$, so $Y = F(X)$, as required.

2. (a) Suppose $g \circ f$ is one-to-one and f is onto. Let $x, y \in B$ with $g(x) = g(y)$. Since f is onto, there exist $a, b \in A$ with $f(a) = x$ and $f(b) = y$. Then $g(f(a)) = g(x) = g(y) = g(f(b))$, i.e. $(g \circ f)(a) = (g \circ f)(b)$, so since $g \circ f$ is one-to-one we have $a = b$, so $f(a) = f(b)$, i.e. $x = y$.

(b) Suppose $g \circ f$ is onto and g is one-to-one. Let $b \in B$. Then $g(b) \in C$, and $g \circ f$ is onto, so there is some $a \in A$ with $(g \circ f)(a) = g(b)$, i.e. $g(f(a)) = g(b)$. Hence, since g is one-to-one, $f(a) = b$ as required.

3. We use the rules $x \in f^{-1}(T) \iff f(x) \in T$, $x \in \bigcap_{\alpha \in \Lambda} T_\alpha \iff (\forall \alpha \in \Lambda)(x \in T_\alpha)$ and $x \in \bigcup_{\alpha \in \Lambda} T_\alpha \iff (\exists \alpha \in \Lambda)(x \in T_\alpha)$.

(a) We have

$$\begin{aligned} x \in f^{-1}\Big(\bigcap_{\alpha \in \Lambda} S_\alpha\Big) &\iff f(x) \in \bigcap_{\alpha \in \Lambda} S_\alpha \\ &\iff (\forall \alpha \in \Lambda)(f(x) \in S_\alpha) \\ &\iff (\forall \alpha \in \Lambda)(x \in f^{-1}(S_\alpha) \\ &\iff x \in \bigcap_{\alpha \in \Lambda} f^{-1}(S_\alpha), \end{aligned}$$

so $f^{-1}(\bigcap_{\alpha \in \Lambda} S_\alpha) = \bigcap_{\alpha \in \Lambda} f^{-1}(S_\alpha)$.

(b) We have

$$\begin{aligned} x \in f^{-1}\Big(\bigcup_{\alpha \in \Lambda} S_\alpha\Big) &\iff f(x) \in \bigcup_{\alpha \in \Lambda} S_\alpha \\ &\iff (\exists \alpha \in \Lambda)(f(x) \in S_\alpha) \\ &\iff (\exists \alpha \in \Lambda)(x \in f^{-1}(S_\alpha) \\ &\iff x \in \bigcup_{\alpha \in \Lambda} f^{-1}(S_\alpha), \end{aligned}$$

so $f^{-1}(\bigcup_{\alpha \in \Lambda} S_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(S_\alpha)$.

4. Let $(*)$ denote the property

$$x \prec_A y \iff f(x) \prec_B f(y) \tag{$*$}$$

and let SOP abbreviate "strictly order preserving". We must prove two implications:

SOP implies one-to-one and $(*)$: Suppose f is SOP. As proved in lectures, f is one-to-one. We must show that $(*)$ holds. So let $x, y \in A$. If $x \prec_A y$ then $x \preceq_A y$ and $x \neq y$, so $f(x) \preceq_B f(y)$ (by SOP) and $f(x) \neq f(y)$ (since f is one-to-one), so $f(x) \prec_B f(y)$. Conversely, if $f(x) \prec_B f(y)$ then $f(x) \preceq_B f(y)$ and $f(x) \neq f(y)$, so $x \preceq_A y$ (by SOP) and $x \neq y$ (since $x = y$ implies $f(x) = f(y)$ for any function), so $x \prec_A y$.

one-to-one and $(*)$ implies SOP: Suppose f is one-to-one and $(*)$ holds. We must show that f is SOP. So let $x, y \in A$. If $x \preceq_A y$ then $x \prec_A y$ or $x = y$, so $f(x) \prec_B f(y)$ (by $(*)$) or $f(x) = f(y)$, so $f(x) \preceq_B f(y)$. Conversely, if $f(x) \preceq_B f(y)$ then $f(x) \prec_B f(y)$ or $f(x) = f(y)$, so $x \prec_A y$ (by $(*)$) or $x = y$ (since f is one-to-one), so $x \preceq_A y$.

5. We prove this by induction. Let P_n be the statement that $n \leq f(n)$.

Base case: We have $1 \leq m$ for all $m \in \mathbb{N}$, so $1 \leq f(1)$, so P_1 is true.

Inductive step: let $n \in \mathbb{N}$, and suppose P_n is true. Then we have $n \leq f(n)$. We also have $n < n+1$, so $f(n) < f(n+1)$. So we have $n < f(n+1)$. But for $m, k \in N$ we have $m < k \iff m+1 \leq k$, so $n < f(n+1)$ implies that $n + 1 \leq f(n + 1)$, so P_{n+1} is true.

Hence, by induction, P_n is true for all $n \in \mathbb{N}$.