1. To show that $\preceq$ is a partial order we must show that it is reflexive, antisymmetric and transitive.

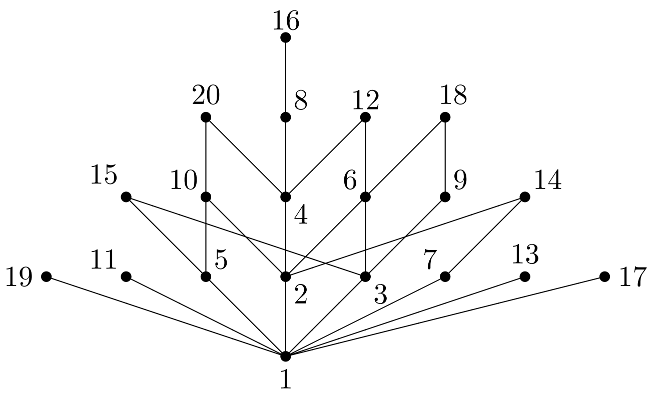
Reflexive: Let $x \in \mathbb{N}$. Then $x = x$ so $x \preceq x$.

Antisymmetric: Let $x, y \in \mathbb{N}$ with $x \preceq y$ and $y \preceq x$. Suppose, for a contradiction, that $x \neq y$. Then we must have $x^2 \leq y$ and $y^2 \leq x$. Since $n \leq n^2$ for all $n \in \mathbb{N}$ we have $x \leq x^2 \leq y \leq y^2 \leq x$, so $x = y$, a contradiction. So we must have $x = y$.

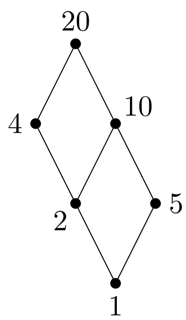
Tranisitive: Let $x, y \in \mathbb{N}$ with $x \preceq y$ and $y \preceq z$. If $x = y$ then (since $y \preceq z$) we have $x \preceq z$, and similarly if $y = z$ then $x \preceq y = z$ so $x \preceq z$. So suppose that $x^2 \leq y$ and $y^2 \leq z$. Since $y \leq y^2$ we have $x^2 \leq y \leq y^2 \leq z$, so $x^2 \leq z$, so $x \preceq z$, as required.

To show that $\preceq$ is not a total order, we exhibit a counterexample: we have $2 \neq 3$ and $2^2 \not\leq 3$ and $3^2 \not\leq 2$ so $2 \not\preceq 3$ and $3 \not\preceq 2$.

2. (a) We have the lattice diagrams



and



(b) The least upper bound for $\{1, 2, 5\}$ in B is 10.

(c) There are a number of choices. One would be $\{2, 3, 5\}$: any upper bound for this set would have to be divisible by 2, 3 and 5, so would have to be at least 30. Another choice would be to take any two maximal elements, say $\{12, 18\}$.

3. Since S has a lower bound, b_0 say, we have $b_0 \in L_S$ so $L_S \neq \emptyset$. Since $S \neq \emptyset$, there is some $s_0 \in S$. Now, for every $b \in L_S$ we have $b \preceq s$ for all $s \in S$, and in particular $b \preceq s_0$. So s_0 is an upper bound for L_S, so L_S has at least one upper bound.

Let $g = \sup L_S$. We must show that g is a lower bound for S, i.e. that $g \preceq s$ for all $s \in S$. So let $s \in S$. As above, for any $b \in B$ we must have $b \preceq s$. Thus s is an upper bound for L_S. Since g is the **least** upper bound for L_S, we must have $g \preceq s$. Since this holds for all $s \in S$, g is a lower bound for S, as required.

Finally, we must show that g is a **greatest** lower bound. So let b be a lower bound for S. Then $b \in L_S$, so (since g is an upper bound for L_S) $b \preceq g$, as required.

4. We must show that ρ is reflexive, antisymmetric and transitive.

Reflexive: Let $(x, y) \in \mathbb{R}^2$. Then $x^2 + y^2 = x^2 + y^2$, so $(x, y)\ \rho\ (x, y)$.

Symmetric: Let $(x, y), (u, v) \in \mathbb{R}^2$ with $(x, y)\ \rho\ (u, v)$. Then $x^2+y^2 = u^2+v^2$, so $u^2+v^2 = x^2+y^2$, so $(u, v)\ \rho\ (x, y)$.

Transitive: Let $(x, y), (u, v), (z, w) \in \mathbb{R}^2$ with $(x, y)\ \rho\ (u, v)$ and $(u, v)\ \rho\ (z, w)$. Then $x^2 + y^2 = u^2 + v^2$ and $u^2 + v^2 = z^2 + w^2$, so $x^2 + y^2 = z^2 + w^2$, i.e. $(x, y)\ \rho\ (z, w)$.

Notice that $(x, y)\ \rho\ (u, v)$ iff $\sqrt{x^2 + y^2} = \sqrt{u^2 + v^2}$, i.e. iff (x, y) and (u, v) are the same distance from the origin $(0, 0)$. Thus $T_{(x,y)}$ is the circle centred at the origin of radius $r = \sqrt{x^2 + y^2}$. In the special case where $(x, y) = (0, 0)$ we have $T_{(0,0)} = \{(0, 0)\}$.