1. Let P_n be the statement $3 \mid n^3 + 2n$.

Base case: $(n = 1)$ $1^3 + 2 \cdot 1 = 3 = 3 \cdot 1$, so $3 \mid 1^3 + 2 \cdot 1$, i.e. P_1 is true.

Inductive step: Let $n \in \mathbb{N}$, and suppose that P_n is true. Then $n^3 + 2n = 3a$ for some $a \in \mathbb{Z}$. So

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2(n+1) \\ &= (n^3 + 2n) + 3n^2 + 3n + 3 \\ &= 3(a + n^2 + n + 1) && \text{(by ind. hyp.)}\end{aligned}$$

hence P_{n+1} is true.

Hence, by induction, P_n is true for all $n \in \mathbb{N}$.

2. Let P_n be the statement $n! > 2^n$.

Base case: $(n = 4)$. We have $4! = 24$ and $2^4 = 16$, so $4! > 2^4$, i.e. P_4 is true.

Inductive step: Let $n \in \mathbb{N}$ with $n \geq 4$, and suppose P_n is true. Then $n! > 2^n$, so

$$\begin{aligned}(n+1)! &= (n+1)n! \\ &> (n+1)2^n && \text{(by ind. hyp.)} \\ &\geq 5 \cdot 2^n && \text{(since } n \geq 4) \\ &> 2 \cdot 2^n \\ &= 2^{n+1},\end{aligned}$$

so $(n+1)! > 2^{n+1}$, i.e. P_{n+1} is true.

Hence, by induction, P_n is true for all $n \geq 4$.

3. If $n = 1$ then $m = mn + 0$, so there is no work to be done. So we will assume that $n > 1$. Let P_m be the statement "There exist integers q and r such that $m = qn + r$ and $0 \leq r < n$".

Base case: $(m = 1)$. We have $1 = 0n + 1$, and $0 \leq 1 < n$, so P_1 is true.

Inductive step: Suppose $m \in \mathbb{N}$ and P_j is true for $1 \leq j \leq m$. We consider three cases: $m+1 < n$, $m + 1 = n$ or $m + 1 > n$.

Case 1: if $m + 1 < n$ then we have $m + 1 = 0n + (m + 1)$ and $0 \leq m + 1 < n$.

Case 2: if $m + 1 = n$ then we have $m + 1 = 1n + 0$ and $0 \leq 0 < n$.

Case 3: if $m + 1 > n$, put $j = (m + 1) - n$. Then $1 \leq j \leq m$, so by the ind. hyp. there exist $q_j, r_j \in \mathbb{Z}$ with $0 \leq r_j < n$ and $j = q_j n + r_j$. But then $m + 1 = j + n = (q_j n + r_j) + n = (q_j + 1)n + r_j$.

So, whichever case holds, P_{m+1} is true.

Hence, by complete induction, P_m is true for all $m \in \mathbb{N}$.

4. The plan is this: first we show that if f is a wibble function then $f(m+2k) = f(m) + 2k$ for all $k \in \mathbb{N}$, by induction on k. We will then extend this to showing that $f(m+2k) = f(m+2k)$ for all $k \in \mathbb{Z}$. Finally, we apply this to f and g: any $n \in \mathbb{Z}$ is either of the form $0 + 2k$ for some $k \in \mathbb{Z}$ or of the form $15 + 2k$ for some k.

For $k \in \mathbb{Z}$, let P_k be the statement "If f is a wibble function and $n \in \mathbb{Z}$ then $f(n+2k) = f(n)+2k$". We first prove by induction that P_k is true for all $k \in \mathbb{N}$.

Base case: if f is a wibble function and $n \in \mathbb{Z}$ then $f(n+2) = f(n) + 2$, by definition of a wibble function.

Inductive step: Let $k \in \mathbb{N}$ and suppose P_k is true. Let f be a wibble function and let $n \in \mathbb{Z}$. Then

$$\begin{aligned} f(n+2(k+1)) &= f((n+2k)+2) \\ &= f(n+2k)+2 && \text{(since } f \text{ is a wibble function)} \\ &= (f(n)+2k)+2 && \text{(by ind. hyp.)} \\ &= f(n)+2(k+1). \end{aligned}$$

Thus P_{k+1} is true.

Hence, by induction, P_k is true for $k \in \mathbb{N}$.

We will now extend this to showing P_k is true for all $k \in \mathbb{Z}$. Note that we P_0 is true, since for any $n \in \mathbb{Z}$ we have $f(n+0) = f(n) + 0$. Finally, we must deal with negative values of k. If $k < 0$ then $-k \in \mathbb{N}$. Put $j = -k$. By the previous part we know that P_j holds. Let f be a wibble function and let $n \in \mathbb{Z}$. Put $m = n+2k$, so $m = n-2j$, so $n = m+2j$. Since P_j is true, $f(m+2j) = f(m)+2j$. In other words we have $f(n) = f(n-2j)+2j$, so $f(n-2j) = f(n)-2j$, i.e. $f(n+2k) = f(n)+2k$.

Thus we have $f(n+2k) = f(n) + 2k$ for all $k \in \mathbb{Z}$.

Now suppose f and g are both wibble functions. Let $n \in \mathbb{Z}$. If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$ so

$$f(n) = f(0+2k) = f(0) + 2k = g(0) + 2k = g(0+2k) = g(n).$$

If n is odd then $n = 2k+1$ for some $k \in \mathbb{Z}$, so $n = 2(k-7)+15$, so $n = 15+2j$ where $j = k-7$. Then

$$f(n) = f(15+2j) = f(15) + 2j = g(15) + 2j = g(15+2j) = g(n).$$

So either way we have $f(n) = g(n)$, as required.

5. (a) ρ is reflexive. Let $x \in \mathbb{Z}$. Then $x^2 \le x^2$ so $x \,\rho\, x$.

(b) ρ is not symmetric: for example we have $2^2 \le 3^2$ but $3^2 \not\le 2^2$, so $2 \,\rho\, 3$ but $3 \not\rho\, 2$.

(c) ρ is not antisymmetric: for example we have $2^2 \le (-2)^2$ and $(-2)^2 \le 2^2$, so $2 \,\rho\, {-2}$ and $-2 \,\rho\, 2$, but $2 \ne -2$.

(d) ρ is transitive. Let $x, y, z \in \mathbb{Z}$ with $x \,\rho\, y$ and $y \,\rho\, z$. Then $x^2 \le y^2 \le z^2$, so $x^2 \le z^2$, so $x \,\rho\, z$.