1. (a) Suppose $m < n$. Then (since $m, n \geq 0$) $m^2 < n^2$, so $m^2 + m < n^2 + n$, i.e. $f(m) < f(n)$.

(b) Suppose $m \not< n$. Then $n \leq m$. If $n < m$ then, by (a), $f(m) < f(n)$, and if $m = n$ then $f(m) = f(n)$. So we have $f(n) \leq f(m)$, so $f(m) \not< f(n)$. Hence, by contraposition, if $f(m) < f(n)$ then $m < n$.

(c) Let $m, n \in \mathbb{N}$. [We must show that if $f(m) = f(n)$ then $m = n$.] Suppose, for a contradiction, that $f(m) = f(n)$ but $m \neq n$. Since $m \neq n$ we have $m < n$ or $n < m$. So, by (a), we have $f(m) < f(n)$ or $f(n) < f(m)$. Either way, we have $f(m) \neq f(n)$, contradicting our assumption that $f(m) = f(n)$. Hence if $f(m) = f(n)$ then $m = n$, in other words f is one-to-one.

2. We need to show existence and uniqueness.

Existence: Put $x_0 = \frac{k-c}{m}$. Note that division by m is allowed because $m \neq 0$. Then $mx_0 + c = m\frac{k-c}{m} + c = (k - c) + c = k$. Thus x_0 is a solution of the equation $mx + c = k$.

Uniqueness: Suppose x and y are both solutions of the equation. Then $mx+c = k$ and $my+c = k$, so $mx + c = my + c$, so (subtracting c from both sides) $mx = my$, so (dividing both sides by m, which is allowed because $m \neq 0$) $x = y$, as required.

[Notice that what you would normally do to solve the equation amounts to showing that **if** x is a solution then $x = \frac{k-c}{m}$: this gives us the uniqueness part (if x and y are both solutions then $x = \frac{k-c}{m}$ and $y = \frac{k-c}{m}$ so $x = y$). It does **not** establish that value of x we found really is a solution. To see this, compare that "solution" with the following "solution" of the equation $x = x+1$: squaring both sides gives $x^2 = (x+1)^2 = x^2+2x+1$, subtracting x^2 from both sides gives $0 = 2x+1$, so $-1 = 2x$ so $x = -\frac{1}{2}$. This "solution" shows that $-\frac{1}{2}$ is the only possible solution of $x = x + 1$, but of course it does not show that $-\frac{1}{2}$ is a solution.]

3. Let S be the set $\{1, 2, 3\}$ and let A be the set $\{ x^2 : x \in S \}$.

(a) We can describe A as $A = \{1, 4, 9\}$ or as $\{ x : (\exists n \in \{1, 2, 3\})(x = n^2) \}$.

(b)
(i) $1 \in S$: True.
(ii) $1 \subseteq S$: False.
(iii) $1 \in A$: True.
(iv) $1 \subseteq A$: False.
(v) $S \in A$: False.
(vi) $S \subseteq A$: False.
(vii) $A \in S$: False.
(viii) $A \subseteq S$: False.

4. We will prove (1) $\implies$ (2), (2) $\implies$ (3) and (3) $\implies$ (1).

(1) $\implies$ (2)**:** Suppose $A \cap B = A$. [We will show that $A \cup B = B$.] Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$ then $x \in A \cap B$ (since $A \cap B = A$) so $x \in B$ in this case also. So, either way, we have $x \in B$. Hence $A \cup B \subseteq B$. Conversely, let $y \in B$. Then $y \in A$ or $y \in B$, so $y \in A \cup B$. Hence $B \subseteq A \cup B$. Combining these we have $A \cup B = B$.

(2) $\implies$ (3)**:** Suppose that $A \cup B = B$. [We will show that $A \subseteq B$.] Let $x \in A$. Then $x \in A$ or $x \in B$, so $x \in A \cup B$, so (since $A \cup B = B$) $x \in B$. Hence $A \subseteq B$.

(3) $\implies$ (1)**:** Suppose that $A \subseteq B$. [We will show that $A \cap B = A$.] Let $x \in A \cap B$. Then $x \in A$ and $x \in B$: in particular $x \in A$. Hence $A \cap B \subseteq A$. Conversely, let $y \in A$. Then (since $A \subseteq B$) we also have $y \in B$, so $y \in A \cap B$. Hence $A \subseteq A \cap B$. Combining these we have $A \cap B = A$.

5. Method 1: Let $x \in A \setminus \bigcup_{\alpha\in\Lambda} B_\alpha$. Then $x \in A$ and $x \notin \bigcup_{\alpha\in\Lambda} B_\alpha$. For every $\alpha \in \Lambda$, we have $x \notin B_\alpha$ (since if it was in any one of these sets then it would be in the union), so $x \in A \setminus B_\alpha$. Since this is true for all $\alpha \in \Lambda$, and $\Lambda \neq \emptyset$, $x \in \bigcap_{\alpha\in\Lambda}(A \setminus B_\alpha)$. Thus $A \setminus \bigcup_{\alpha\in\Lambda} B_\alpha \subseteq \bigcap_{\alpha\in\Lambda}(A \setminus B_\alpha)$. Conversely, let $y \in \bigcap_{\alpha\in\Lambda}(A \setminus B_\alpha)$. Since there is at least one $\alpha_0 \in \Lambda$, and we have $y \in A \setminus B_{\alpha_0}$, we certainly have $y \in A$. Also, for every $\alpha \in \Lambda$ we have $y \in A \setminus B_\alpha$, so $y \notin B_\alpha$. Since there is no α with $y \in B_\alpha$, we have $y \notin \bigcup_{\alpha\in\Lambda} B_\alpha$. Hence $y \in A \setminus \bigcup_{\alpha\in\Lambda} B_\alpha$. Thus $\bigcap_{\alpha\in\Lambda}(A \setminus B_\alpha) \subseteq A \setminus \bigcup_{\alpha\in\Lambda} B_\alpha$. Combining these we have $A \setminus \bigcup_{\alpha\in\Lambda} B_\alpha = \bigcap_{\alpha\in\Lambda}(A \setminus B_\alpha)$.

Method 2: For any x we have

$$\begin{aligned}
x \in A \setminus \bigcup_{\alpha\in\Lambda} B_\alpha &\iff x \in A \wedge x \notin \bigcup_{\alpha\in\Lambda} B_\alpha \\
&\iff x \in A \wedge \sim\!(x \in \bigcup_{\alpha\in\Lambda} B_\alpha) \\
&\iff x \in A \wedge \sim\!(\exists \alpha \in \Lambda)(x \in B_\alpha) \\
&\iff x \in A \wedge (\forall \alpha \in \Lambda)(x \notin B_\alpha) \\
&\iff (\forall \alpha \in \Lambda)(x \in A \wedge x \notin B_\alpha) \qquad \text{(see below)} \\
&\iff (\forall \alpha \in \Lambda)(x \in A \setminus B_\alpha) \\
&\iff x \in \bigcap_{\alpha\in\Lambda}(A \setminus B_\alpha)
\end{aligned}$$

All but one of these steps is obviously an equivalence. The implication

$$x \in A \wedge (\forall \alpha \in \Lambda)(x \notin B_\alpha) \implies (\forall \alpha \in \Lambda)(x \in A \wedge x \notin B_\alpha)$$

is obvious but the converse is only true because Λ is not empty. [In general, a statement $(\forall x \in S)P(x)$ could be true because S is empty and not because there actually is any x at all which makes $P(x)$ true—for example the statement "All of my aeroplanes have five wings" is true because I don't own any aeroplanes at all.]