1. (a) Suppose (s_n) converges to L. Let $\varepsilon > 0$. Put $\eta = \frac{\varepsilon}{2}$. Choose $N \in \mathbb{N}$ so that if $n > N$ then $|s_n - L| < \eta$. Let $m, n > N$. Then

$$\begin{aligned} |s_m - s_n| &= |s_m - L + L - s_n| \\ &\leq |s_m - L| + |L - s_n| && \text{(triangle inequality)} \\ &= |s_m - L| + |s_n - L| \\ &< \eta + \eta \\ &= \varepsilon, \end{aligned}$$

as required. Thus (s_n) is a Cauchy sequence.

(b) Suppose (s_n) is a Cauchy sequence. Putting $\varepsilon = \frac{1}{2}$, we can find N such that for all $m, n > N$, $|s_m - s_n| < \frac{1}{2}$. In particular, if $m > N$ then $|s_m - s_{N+1}| < \frac{1}{2}$, so $|s_m| < |s_{N+1}| + \frac{1}{2}$. Thus, putting $b = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s_{N+1}| + \frac{1}{2}\}$ we have $|s_m| < b$ for all $m \in \mathbb{N}$, so (s_n) is bounded above by b and below by $-b$, so (s_n) is bounded.

2. $f + g$**:** Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Put $\eta = \frac{\varepsilon}{2}$. Choose $\delta_1 > 0$ such that if $|y - x| < \delta_1$ then $|f(y) - f(x)| < \eta$, and $\delta_2 > 0$ such that if $|y - x| < \delta_2$ then $|g(y) - g(x)| < \eta$. Put $\delta = \min\{\delta_1, \delta_2\}$. Let $y \in \mathbb{R}$ with $|y - x| < \delta$. Then

$$\begin{aligned} |(f+g)(y) - (f+g)(x)| &= |f(y) + g(y) - f(x) - g(x)| \\ &= |(f(y) - f(x)) + (g(y) - g(x))| \\ &\leq |f(y) - f(x)| + |g(y) - g(x)| && \text{(triangle inequality)} \\ &< \eta + \eta = \varepsilon, \end{aligned}$$

so $|(f+g)(y) - (f+g)(x)| < \varepsilon$ as required. Thus $f + g$ is continuous.

fg**:** Let $x \in \mathbb{R}$, $\varepsilon > 0$. Put $\eta = \frac{\varepsilon}{|f(x)| + \frac{1}{2} + |g(x)|}$. Choose $\delta_1 > 0$ such that if $|y - x| < \delta_1$ then $|f(y) - f(x)| < \eta$, and $\delta_2 > 0$ such that if $|y - x| < \delta_2$ then $|g(y) - g(x)| < \min\{\eta, \frac{1}{2}\}$. Put $\delta = \min\{\delta_1, \delta_2\}$. Let $y \in \mathbb{R}$ with $|y - x| < \delta$. Then

$$\begin{aligned} |(fg)(y) - (fg)(x)| &= |f(y)g(y) - f(x)g(x)| \\ &= |f(y)g(y) - f(x)g(y) + f(x)g(y) - f(x)g(x)| \\ &\leq |(f(y) - f(x))g(y)| + |f(x)(g(y) - g(x)| && \text{(triangle inequality)} \\ &= |f(y) - f(x)||g(y)| + |f(x)||g(y) - g(x)| \\ &\leq |f(y) - f(x)|(|g(x)| + \tfrac{1}{2}) + |f(x)||g(y) - g(x)| \\ & && \text{(since } |g(y)| < |g(x) + \tfrac{1}{2}\text{)} \\ &< \eta(|g(x)| + \tfrac{1}{2}) + |f(x)|\eta \\ &= \varepsilon, \end{aligned}$$

so $|(fg)(y) - (fg)(x)| < \varepsilon$, as required. Thus fg is continuous.

$f \circ g$**:** Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Since f is continuous, it is continuous at $g(x)$. Choose $\eta > 0$ so that if $|y - g(x)| < \eta$ then $|f(y) - f(g(x))| < \varepsilon$. Choose $\delta > 0$ such that if $|y - x| < \delta$ then $|g(y) - g(x)| < \eta$. Let $y \in \mathbb{R}$ with $|y - x| < \delta$. Then $|g(y) - g(x)| < \eta$, so $|f(g(y)) - f(g(x))| < \varepsilon$, i.e. $|(f \circ g)(y) - (f \circ g)(x)| < \varepsilon$, as required. Thus $f \circ g$ is continuous.

3. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Put $\eta = \frac{\varepsilon}{|a|}$. Since f is continuous at bx, we can choose $\delta_1 > 0$ so that if $|y - bx| < \delta$ then $|f(y) - f(bx)| < \eta$. Put $\delta = \frac{\delta_1}{|b|}$. Let $y \in \mathbb{R}$ with $|y - x| < \delta$. Then

$$\begin{aligned} |g(y) - g(x)| &= |af(by) - af(bx)| \\ &= |a||f(by) - f(bx)| \\ &< |a|\eta & \text{(because } |by - bx| = |b||y - x| < |b|\delta = \delta_1) \\ &= \varepsilon, \end{aligned}$$

as required. Thus g is continuous.

4. When showing that three statements are equivalent, there are six implications to prove, but we usually only prove three or four of them, for example (1) $\implies$ (2), (2) $\implies$ (3) and (3) $\implies$ (1), or (1) $\implies$ (2), (2) $\implies$ (1), (2) $\implies$ (3) and (3) $\implies$ (2). When proving four statements are equivalent, there are twelve implications to prove, and many more choices of which four or five or six to establish in order to imply all the others. In this case, we will prove (1) $\implies$ (2), (2) $\implies$ (3), (3) $\implies$ (2), (2) $\implies$ (1), (1) $\implies$ (4) and (4) $\implies$ (1).

(1) $\implies$ (2)**:** Proved in lectures (week 11, Lemma 20).

(2) $\implies$ (3)**:** Suppose that for all open sets U, $f^{-1}(U)$ is open. Let $C \subseteq \mathbb{R}$ be closed. We wish to show that $f^{-1}(C)$ is closed, i.e. that $\mathbb{R} \setminus f^{-1}(C)$ is open. We know that $\mathbb{R} \setminus C$ is open, so by (2) it is enough to show that $f^{-1}(\mathbb{R} \setminus C) = \mathbb{R} \setminus f^{-1}(C)$. [In fact, this is true for any set C, whether or not C is closed: we will use this again shortly.]

Let $x \in f^{-1}(\mathbb{R} \setminus C)$. Then $x \in \mathbb{R}$ and $f(x) \in \mathbb{R} \setminus C$, so $f(x) \notin C$, so $x \notin f^{-1}(C)$, so $x \in \mathbb{R} \setminus f^{-1}(C)$. Thus $f^{-1}(\mathbb{R} \setminus C) \subseteq \mathbb{R} \setminus f^{-1}(C)$.

Conversely, let $y \in \mathbb{R} \setminus f^{-1}(C)$. Then $y \notin f^{-1}(C)$, so $f(y) \notin C$, so $f(y) \in \mathbb{R} \setminus C$, so $y \in f^{-1}(\mathbb{R} \setminus C)$. Thus $f^{-1}(C) \subseteq f^{-1}(\mathbb{R} \setminus C)$.

Hence $f^{-1}(\mathbb{R} \setminus C) = \mathbb{R} \setminus f^{-1}(C)$, as required.

(3) $\implies$ (2)**:** Suppose $f^{-1}C$ is closed for all closed $C \subseteq \mathbb{R}$. Let $U \subseteq \mathbb{R}$ be open. Then $\mathbb{R} \setminus U$ is closed, so $f^{-1}(\mathbb{R} \setminus U)$ is closed, in other words $\mathbb{R} \setminus f^{-1}(\mathbb{R} \setminus U)$ is open. As above, we know that $f^{-1}(\mathbb{R} \setminus U) = \mathbb{R} \setminus f^{-1}(U)$, so $\mathbb{R} \setminus (\mathbb{R} \setminus f^{-1}(U))$ is open, i.e. $f^{-1}(U)$ is open, as required.

(2) $\implies$ (1)**:** Suppose $f^{-1}(U)$ is open for all open sets $U \subseteq \mathbb{R}$. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. We will show that $B_\varepsilon(f(x))$ is open. So let $y \in B_\varepsilon(f(x))$. Then $|y - f(x)| < \varepsilon$. Put $\eta = \varepsilon - |y - f(x)|$. Let $z \in B_\eta(y)$. By the triangle inequality, $|z - f(x)| \leq |z - y| + |y - f(x)| < \eta + |y - f(x)| = \varepsilon$, so $z \in B_\varepsilon(f(x))$. In other words, for every $y \in B_\varepsilon(f(x))$ there is some $\eta > 0$ (namely $\eta = \varepsilon - |y - f(x)|$) with $B_\eta(y) \subseteq B_\varepsilon(f(x))$. Thus $B_\varepsilon(f(x))$ is open, so by (2) $f^{-1}(B_\varepsilon(f(x)))$ is open. Now $x \in f^{-1}(B_\varepsilon(f(x)))$ (since $f(x) \in B_\varepsilon(f(x))$), so there is some $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(x))$. Let $y \in \mathbb{R}$ with $|y - x| < \delta$. Then $y \in B_\delta(x)$, so $y \in f^{-1}(B_\varepsilon(x))$, so $f(y) \in B_\varepsilon(f(x))$, so $|f(y) - f(x)| < \varepsilon$, as required.

(1) $\implies$ (4)**:** Proved in lectures (week 11, Lemma 21).

(4) $\implies$ (1)**:** Suppose that for all sequences (s_n), if $s_n \to a$ as $n \to \infty$ then $f(s_n) \to f(a)$ as $n \to \infty$. Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. Suppose, for a contradiction, that there is no $\delta > 0$ such that $|y - x| < \delta$ implies that $|f(y) - f(x)| < \varepsilon$. In particular, for each $n \in \mathbb{N}$ we have $\frac{1}{n} > 0$ so there is some y_n with $|y_n - x| < \frac{1}{n}$ but $|f(y_n) - f(x)| \geq \varepsilon$. But then $y_n \to x$ as $n \to \infty$ but $f(y_n) \nrightarrow f(x)$ as $n \to \infty$, contradicting (4). Thus f must be continuous.

5. Let $x \in \mathbb{R}$, and let $h \in \mathbb{R}$ with $h \neq 0$. Then

$$\begin{aligned}
\frac{(fg)(x+h)-(fg)(x)}{h} &= \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\
&= \frac{f(x+h)-f(x)}{h}g(x+h)+f(x)\frac{g(x+h)-g(x)}{h}
\end{aligned}$$

Now $\lim_h \to 0\frac{f(x+h)-f(x)}{h} = f'(x)$, $\lim_{h\to 0} g(x+h) = g(x)$, and $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$, so the limit of the right hand side equals $f'(x)g(x) + f(x)g'(x)$. Hence the limit of the left hand side as $h \to 0$ also exists and equals the same, in other words $(fg)'(x)$ exists and equals $f'(x)g(x)+f(x)g'(x)$.