1. We must show existence and uniqueness of the solution.

Existence: Put $x = \frac{a}{b} = a \cdot \frac{1}{b}$. Then $bx = b \cdot a \cdot \frac{1}{b} = a \cdot b \cdot \frac{1}{b} = a \cdot 1_F = a$, so x is a solution of the equation $a = bx$.

Uniqueness: Suppose x and y are both solutions, in other words we have $a = bx$ and $a = by$. Then $bx = by$, so $\frac{1}{b}bx = \frac{1}{b}by$, so $1_F x = 1_F y$, so $x = y$.

2. (a) Suppose $a < b$. Then $b - a \in P$. Now $(b+c)-(a+c) = (b-a)+(c-c) = (b-a)-0_F = b-a$, so $(b+c)-(a+c) \in P$, so $a + c < b + c$.

(b) Suppose $a \leq b$. Then $a < b$ or $a = b$. If $a < b$ then $a + c < b + c$ by part (a), and if $a = b$ then $a + c = b + c$. So either way we have $a + c \leq b + c$.

(c) Suppose $a < b$ and $0_F < c$. Then $b - a \in P$ and $c - 0_F \in P$, i.e. $c \in P$, so $(b - a)c \in P$, i.e. $bc - ac \in P$, so $ac < bc$.

(d) Suppose, for a contradiction that $a \in P$ but $\frac{1}{a} \notin P$. Then we must have $\frac{1}{a} = 0_F$ or $-\frac{1}{a} \in P$.

Case 1: Suppose $\frac{1}{a} = 0$. Then $1_F = a \cdot \frac{1}{a} = a \cdot 0_F = 0_F$, so $1_F = 0_F$. But the field axioms specify that $1_F \neq 0_F$.

Case 2: Suppose $-\frac{1}{a} \in P$. Then $a(-\frac{1}{a}) \in P$, so $-(a \cdot \frac{1}{a}) \in P$, i.e. $-1_F \in P$, so $1_F \notin P$. But this contradicts a result proved in lectures that $1_F \in P$.

So neither case 1 nor case 2 is possible, so we must have $\frac{1}{a} \in P$ as required.

(e) Suppose $0_F < a < b$. Then $b - a \in P$ and $a, b \in P$, so by part (d) we have $\frac{1}{a} \in P$ and $\frac{1}{b} \in P$. Thus $\frac{1}{a} \cdot \frac{1}{b} \in P$, so $(b-a)\frac{1}{a}\frac{1}{b} \in P$. But $(b-a)\frac{1}{a}\frac{1}{b} = b\frac{1}{b}\frac{1}{a} - a\frac{1}{a}\frac{1}{b} = \frac{1}{a} - \frac{1}{b}$, so $\frac{1}{a} - \frac{1}{b} \in P$, so $\frac{1}{b} < \frac{1}{a}$. Since we also have $\frac{1}{b} \in P$ we have $0_F < \frac{1}{b} < \frac{1}{a}$ as required.

3. (a) Suppose first that $[c, d] \subseteq [a, b]$. Since $c \leq c \leq d$, $c \in [c, d] \subseteq [a, b]$, so $a \leq c \leq b$. Similarly, $d \in [c, d] \subseteq [a, b]$ so $a \leq d \leq b$. Thus $a \leq c$ and $d \leq b$.

Conversely, suppose $a \leq c$ and $d \leq b$. Let $x \in [c, d]$. Then $c \leq x \leq d$, so $a \leq c \leq x \leq d \leq b$, so $a \leq x \leq b$, so $x \in [a, b]$. Thus $[c, d] \subseteq [a, b]$.

(b) Let $c \in \mathbb{R}$. We have

$$\begin{aligned}
c \in \bigcap_{n\in\mathbb{N}} [a_n, b_n] &\iff (\forall n \in \mathbb{N})(c \in [a_n, b_n]) \\
&\iff (\forall n \in \mathbb{N})(a_n \leq c \leq b_n) \\
&\iff (\forall n \in \mathbb{N})(a_n \leq c) \wedge (\forall n \in \mathbb{N})(c \leq b_n) \\
&\iff c \text{ is an upper bound for } \{\, a_n : n \in \mathbb{N} \,\} \text{ and} \\
&\qquad\qquad c \text{ is a lower bound for } \{\, b_n : n \in \mathbb{N} \,\}
\end{aligned}$$

(c) Let $n \in \mathbb{N}$. Let P_k be the statement "$a_n \le a_{n+k}$ and $b_{n+k} \le b_n$".

Base case: From $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and part (a) we have $a_n \le a_{n+1}$ and $b_{n+1} \le b_n$. So P_1 is true.

Inductive step: Let $k \in \mathbb{N}$, and suppose that P_k is true. As in the base step, we have $a_{n+k} \le a_{n+k+1}$ and $b_{n+k+1} \le b_{n+k}$, so by the inductive hypothesis we have $a_n \le a_{n+k} \le a_{n+k+1}$ and $b_{n+k+1} \le b_{n+k} \le b_n$, so $a_n \le a_{n+k+1}$ and $b_{n+k+1} \le b_n$, i.e. P_{k+1} is true.

Hence, by induction, P_k is true for all $k \in \mathbb{N}$.

(d) Let $m, n \in \mathbb{N}$. We consider three cases: $m < n$, $m = n$ or $n < m$. If $m < n$ then there is some k with $m + k = n$. By (c) we have $a_m \le a_{m+k} = a_n < b_n$, so $a_m \le b_n$. If $m = n$ then we have $a_m = a_n < b_n$, so $a_m \le b_n$. Finally, if $n < m$ then $m = n + k$ for some $k \in \mathbb{N}$, so by part (c) we have $a_m < b_m = b_{n+k} \le b_n$, so $a_m \le b_n$.

(e) Put $S = \{ a_n : n \in \mathbb{N} \}$. Then $a_1 \in S$, so $S \neq \emptyset$, and by part (d) we know that S is bounded above by b_1, so S has a least upper bound. Put $c = \sup S$.

(f) Let $n \in \mathbb{N}$. Then, by part (d), $a_m \le b_n$ for all m, so b_n is an upper bound for S. Thus, since c is the least upper bound for S, $c \le b_n$. Since this holds for all n, c is a lower bound for $\{ b_n : n \in \mathbb{N} \}$.

(g) From (e) and (f), c is both an upper bound for $\{ a_n : n \in \mathbb{N} \}$ and a lower bound for $\{ b_n : n \in \mathbb{N} \}$, so by (b) we have $c \in \bigcap_{n\in\mathbb{N}}[a_n, b_n]$, so $\bigcap_{n\in\mathbb{N}}[a_n, b_n] \neq \emptyset$.

4. Suppose first that $s_n \to L$. Let $\varepsilon > 0$. Then there is some $N \in \mathbb{N}$ such that for all $n > N$, $|s_n - L| < \varepsilon$. Let $n > N$. Then $|a_n - L| = |s_{2n-1} - L| < \varepsilon$, since $2n - 1 \ge n > N$. Thus $a_n \to L$. Similarly, if $n > N$ then $|b_n - L| = |s_{2n} - L| < \varepsilon$ since $2n \ge n > N$. Thus $b_n \to L$.

Conversely, suppose that $a_n \to L$ and $b_n \to L$. Let $\varepsilon > 0$. There exist $N_1, N_2 \in \mathbb{N}$ such that if $n > N - 1$ then $|a_n - L| < \varepsilon$ and if $n > N_2$ then $|b_n - L| < \varepsilon$. Put $N = \max\{2N_1 - 1, 2N_2\}$. Let $n > N$. Then n is either odd or even.

Case 1: Suppose n is odd. Then $n = 2k - 1$ for some $k \in \mathbb{N}$, and since $n > N$, $n > 2N_1 - 1$, i.e. $2k - 1 > 2N_1 - 1$, so $2k > 2N_1$, so $k > N_1$. Thus $|a_k - L| < \varepsilon$, and since $s_n = s_{2k-1} = a_k$ we have $|a_n - L| < \varepsilon$.

Case 2: Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{N}$, and since $n > N$, $n > 2N_2$, i.e. $2k > 2N_2$, so $k > N_2$. Thus we have $|b_k - L| < \varepsilon$, and $s_n = s_{2k} = b_k$, so $|s_n - L| < \varepsilon$.

Thus in either case we have $|s_n - L| < \varepsilon$. Hence $s_n \to L$.