1. We know that $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Adding these two inequalities we get $-|a| - |b| \leq a + b \leq |a| + |b|$, i.e. $-(|a| + |b|) < a + b \leq |a| + |b|$, so $|a + b| \leq |a| + |b|$.

2. Put $a = x - y$, $b = y - z$. Then $a + b = (x - y) + (y - z) = x - z$, so from Question 1 we have $|x - z| = |a + b| \leq |a| + |b| = |x - y| + |y - z|$, i.e. $|x - z| \leq |x - y| + |y - z|$.

What this says is that the distance from x to z is less than or equal to the distance from x to y plus the distance from y to z. Thinking of x, y and z as corners of a triangle, it asserts that it is no further to go directly from x to z than to go from x to y and from there to z.

3. To say that (s_n) does not converge to L is to say that there is some $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there is an $n \in \mathbb{N}$ with $n \geq N$ such that $|s_n - L| > \varepsilon$.

4. Let L and M be limits of the sequence (s_n). Suppose, for a contradiction, that $L \neq M$. Then $|L - M| > 0$. Put $\varepsilon = \frac{|L-M|}{2}$. Then $\varepsilon > 0$, so there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|s_n - L| < \varepsilon$, and there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|s_n - M| < \varepsilon$. Put $N = \max\{N_1, N_2\}$, and put $n = N + 1$. Then $n \geq N_1$ and $n \geq N_2$, so $|s_n - L| < \varepsilon$ and $|s_n - M| < \varepsilon$. But then $|L - s_n| < \varepsilon$, so by the triangle inequality we have

$$\begin{aligned} |L - M| &= |(L - s_n) + (s_n - M)| \\ &\leq |L - s_n| + |s_n - M| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon \\ &= |L - M|, \end{aligned}$$

i.e. $|L - M| < |L - M|$. This is impossible, so we cannot have $L \neq M$, so $L = M$ as required.

5. Let $\varepsilon > 0$. Put $\delta = \frac{\varepsilon}{2}$. Then $\delta > 0$, so there exists $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$, $|a_n - L| < \delta$, and there exists $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$, $|b_n - M| < \delta$. Put $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ with $n \geq N$. Then $n \geq N_1$ and $n \geq N_2$, so $|a_n - L| < \delta$ and $|b_n - M| < \delta$, so

$$\begin{aligned} |c_n - (L + M)| &= |a_n + b_n - L - M| \\ &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \\ &= \delta + \delta \\ &= 2\delta \\ &= \varepsilon, \end{aligned}$$

i.e. $|c_n - (L + M)| < \varepsilon$, as required.