1. Let $a(x) = x^3 - x^2 - 15x - 25$.

(a) $a(5) = 5^3 - 5^2 - 15 \cdot 5 - 25 = 125 - 25 - 75 - 25 = 0$.

(b) From this we know that $x - 5 \mid x^3 - x^2 - 15x - 25$, so we do long division:

$$\begin{array}{r|rrrr} & x^2 & +4x & +5 & \\ \hline x-5 & x^3 & -x^2 & -15x & -25 \\ & x^3 & -5x^2 & & \\ \hline & & 4x^2 & -15x & \\ & & 4x^2 & -20x & \\ \hline & & & 5x & -25 \\ & & & 5x & -25 \\ \hline & & & & 0 \end{array}$$

From this we see that $a(x) = (x-5)(x^2 + 4x + 5)$. Now the solutions of $x^2 + 4x + 5$ are $x = \dfrac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2}$. But $4^2 - 4 \cdot 1 \cdot 5 < 0$, so there are no real solutions. Thus, by the factor theorem, $x^2 + 4x + 5$ has no linear factors so it is irreducible.

2. (a) We need to confirm that $(x * y) * (y^{-1} * x^{-1}) = e$ and $(y^{-1} * x^{-1}) * (x * y) = e$. However, since we know that G is a group we know that if the first of these holds then the second also holds. So we just check the first:

$$\begin{aligned} (x * y) * (y^{-1} * x^{-1}) &= x * (y * (y^{-1} * x^{-1})) \\ &= x * ((y * y^{-1}) * x^{-1}) \\ &= x * (e * x^{-1}) \\ &= x * x^{-1} \\ &= e, \end{aligned}$$

as required.

(b) Suppose first that $x * y = y * x$. By part (a) we know that $(y * x)^{-1} = x^{-1} * y^{-1}$, so substituting in $x * y = y * x$ we get $(x * y)^{-1} = x^{-1} * y^{-1}$.

Conversely, suppose that $(x * y)^{-1} = x^{-1} * y^{-1}$. Then we have

$$\begin{aligned} (x * y)^{-1} &= x^{-1} * y^{-1} \\ (x * y) * (x * y)^{-1} &= (x * y) * (x^{-1} * y^{-1}) \\ e &= (x * y) * (x^{-1} * y^{-1}) \\ e * y &= ((x * y) * (x^{-1} * y^{-1})) * y \\ y &= ((x * y) * x^{-1}) * (y * y^{-1}) \\ y &= (x * y) * x^{-1} \\ y * x &= ((x * y) * x{-}1) * x \\ &= x * y \end{aligned}$$

so we have $x * y = y * x$, as required.

3. **One-to-one:** Let $x, y \in G$ with $f_g(x) = f_g(y)$. Then $g^{-1}xg = g^{-1}yg$, so $g(g^{-1}xg)g^{-1} = g(g^{-1}yg)g^{-1}$, so $exe = eye$, so $x = y$.

Onto: let $b \in G$. [We need an $a \in G$ with $f_g(a) = b$, i.e. with $g^{-1}ag = b$. Rearranging this we get $gg^{-1}ag^{-1}g = gbg^{-1}$, i.e. $a = gbg^{-1}$] Put $a = gbg^{-1}$. Then $f_g(a) = g^{-1}ag = g^{-1}(gbg^{-1})g = ebe = b$, i.e. $f_g(a) = b$ as required.

Preserves $*$: let $x, y \in G$. Then $f_g(x)f_g(y) = (g^{-1}xg)(g^{-1}yg) = g^{-1}xeyg = g^{-1}(xy)g = f_g(xy)$, as required.