1. (a) Let $x \in \mathbb{Z}$. Then we have

$$\begin{aligned} (-x) + x &= x + (-x) && \text{(commutative law)} \\ &= 0 && \text{(definition of } -x) \end{aligned}$$

so by definition of $-(-x)$, $-(-x) = x$.

(b) Let $x \in \mathbb{Z}$. Then we have

$$\begin{aligned} x + (-1) \cdot x &= 1 \cdot x + (-1) \cdot x && \text{(definition of 1)} \\ &= x \cdot 1 + x \cdot (-1) && \text{(commutative law)} \\ &= x \cdot (1 + (-1)) && \text{(distributive law)} \\ &= x \cdot 0 && \text{(definition of } -1) \\ &= 0 \cdot x && \text{(commutative law)} \\ &= 0. && \text{(proved in lectures)} \end{aligned}$$

Hence, by definition of $-x$, $-x = (-1) \cdot x$.

2. Suppose that $a, b, c, x, y \in \mathbb{Z}$ with $c \mid a$ and $c \mid b$. Then there exist $p, q \in \mathbb{Z}$ with $a = cp$ and $b = cq$. But then

$$ax + by = (cp)x + (cq)y = c(px + qy),$$

and $(px + qy) \in \mathbb{Z}$, so $c \mid ax + by$ as required.

3. (a) Suppose that r and $r + s$ are not relatively prime. Then there is some $d > 1$ such that $d \mid r$ and $d \mid r + ss$. But then, by the previous question, $d \mid r \cdot (-1) + (r + s) \cdot 1$, i.e. $d \mid s$. Thus d is a common divisor of r and s, so r and s are not relatively prime. Hence, by contraposition, if r and s are relatively prime then so are r and $r + s$.

(b) Suppose there exist $x, y \in \mathbb{Z}$ with $rx + sy = 1$. Put $d = \gcd(r, s)$. Then, by the previous question, $d \mid 1$, so $d = 1$. So r and s are relatively prime.

4. Let $x, y \in \mathbb{Z}$ with $x + 1 = y + 1$. Suppose, for a contradiction, that $x \neq y$. Then $x < y$ or $y < x$: without loss of generality (WLOG) we have $x < y$. We also have $y < y + 1 = x + 1$. So we have $x < y < x + 1$, contradicting the fact that $x + 1$ is an immediate successor of x. Hence by contradiction we cannot have $x \neq y$, so $x = y$ as required.