1. The problem is that there might be some values of x and y for which $x < y \implies f(x) < f(y)$ and others for which $x < y \implies f(x) > f(y)$. For f to be strictly monotone we would have to get the same answer for all x and y.

To understand the distinction, consider the difference between "Every child at the school is a girl or a boy" with "All the children at the school are girls, or all of them are boys". The second statement asserts that it is a single-sex school, the first merely asserts that all the pupils are human!

2. (a) $\emptyset \in \emptyset$: False
(b) $\emptyset \subseteq \emptyset$: True
(c) $\emptyset \in \{\emptyset\}$: True
(d) $\emptyset \subseteq \{\emptyset\}$: True
(e) $\emptyset = \{\emptyset\}$: False
(f) $\{\emptyset\} \in \emptyset$: False
(g) $\{\emptyset\} \in \{\emptyset\}$: False
(h) $\{\emptyset\} \subseteq \emptyset$: False
(i) $\{\emptyset\} \subseteq \{\emptyset\}$: True

3. (a) $\bigcup_{n\in N} A_n = \mathbb{N}$.
(b) $\bigcap_{n\in N} A_n = \{1\}$.

4. (a) Let $x \in \bigcup_{a\in S}\{a\}$. Then $x \in \{a\}$ for some $a \in S$. But then $x = a$, so $x \in S$. Hence $\bigcup_{a\in S}\{a\} \subseteq S$.
Conversely, let $y \in S$. Then $y \in \{y\}$, so there is some $a \in S$ (namely $a = y$) with $y \in \{a\}$. So $y \in \bigcup_{a\in S}\{a\}$. Hence $\bigcup_{a\in S}\{a\} \subseteq S$.
Combining these we have $\bigcup_{a\in S}\{a\} = S$.

(b) Let $x \in \bigcup_{A\in\mathcal{P}(S)} A$. Then $x \in A$ for some $A \in \mathcal{P}(S)$. Since $A \in \mathcal{P}(S)$, $A \subseteq S$, and $x \in A$, so $x \in S$. Thus $\bigcup_{A\in\mathcal{P}(S)} A \subseteq S$.
Conversely, let $y \in S$. Then $\{y\} \subseteq S$, so $\{y\} \in \mathcal{P}(S)$. So there is some $A \in \mathcal{P}(S)$ (namely $A = \{y\}$) with $y \in A$. So $y \in \bigcup_{A\in\mathcal{P}(S)} A$. Hence $S \subseteq \bigcup_{A\in\mathcal{P}(S)} A$.
Combining these we have $\bigcup_{A\in\mathcal{P}(S)} A = S$.

5. Let $B \in \mathcal{P}(A_{n+1})$. [We want to show that $B \in \mathcal{P}(A_n) \cup \{\, S \cup \{n+1\} : S \in \mathcal{P}(A_n)\,\}$, i.e. that $B \in \mathcal{P}(A_n)$ or $B \in \{\, S \cup \{n+1\} : S \in \mathcal{P}(A_n)\,\}$.] Either $n+1 \in B$ or $n+1 \notin B$. If $n+1 \notin B$ then $B \subseteq A_{n+1} = A_n \cup \{n+1\}$, and $n+1 \notin B$, so $B \subseteq A_n$. Thus in this case $B \in \mathcal{P}(A_n)$. On the other hand, if $n+1 \in B$, put $C = B \setminus \{n+1\}$. Then $C \subseteq A_n$ and so $C \in \mathcal{P}(A_n)$, so $B = C \cup \{n+1\} \in \{\, S \cup \{n+1\} : S \in \mathcal{P}(A_n)\,\}$. So in either case we have $B \in \mathcal{P}(A_n) \cup \{\, S \cup \{n+1\} : S \in \mathcal{P}(A_n)\,\}$. Hence $\mathcal{P}(A_{n+1} \subseteq \mathcal{P}(A_n) \cup \{\, S \cup \{n+1\} : S \in \mathcal{P}(A_n)\,\}$.

Conversely, suppose that $D \in \mathcal{P}(A_n) \cup \{\, S \cup \{n+1\} : S \in \mathcal{P}(A_n)\,\}$. Then $D \in \mathcal{P}(A_n) \cup \{\, S$ or $D \in \{n+1\} : S \in \mathcal{P}(A_n)\,\}$. In the first case we have $D \subseteq A_n \subseteq A_{n+1}$. In the second case, $D = S \cup \{n+1\}$ for some $S \in \mathcal{P}(A_n)$. Let $x \in D$. Then $x \in S \subseteq A_n$, or $x = n+1$. Either way, $x \in A_{n+1}$. Thus $D \subseteq A_{n+1}$. Since this holds in either case, we must have $D \subseteq A_{n+1}$, i.e. $D \in \mathcal{P}(A_{n+1})$. Thus $\mathcal{P}(A_n) \cup \{\, S \cup \{n+1\} : S \in \mathcal{P}(A_n)\,\} \subseteq \mathcal{P}(A_{n+1})$.

Combining these we have $\mathcal{P}(A_{n+1}) = \mathcal{P}(A_n) \cup \{\, S \cup \{n+1\} : S \in \mathcal{P}(A_n)\,\}$.