1. Suppose $x^n = x^{n+k}$. Then $x^n x^k = x^n$, so multiplying by x^{-n} gives $x^k = e$.

Conversely, suppose $x^k = e$. Then $x^{n+k} = x^n x^k = x^n e = x^n$.

2. Suppose G is finite. Let $x \in G$. Then the elements x^n for $n \in \mathbb{N}$ cannot all be different, so there exist $n, m \in \mathbb{N}$ with $n < m$ and $x^n = x^m$. Since $n < m$, there is some k with $n + k = m$. Then $x^n + k = x^n$, so $x^k = e$ by the previous question.

3. We must check that $e \in \langle x \rangle$, that if $g, h \in \langle x \rangle$ then $gh \in \langle x \rangle$, and that if $g \in \langle x \rangle$ then $g^{-1} \in \langle x \rangle$.

- $e = x^0 \in \langle x \rangle$.
- if $g, h \in \langle x \rangle$ then $g = x^n$, $h = x^m$ for some $n, m \in \mathbb{Z}$. But then $gh = x^{m+n} \in \langle x \rangle$.
- if $g \in \langle x \rangle$ then $g = x^n$ for some $n \in \mathbb{Z}$, so $g^{-1} = (x^n)^{-1} = x^{-n} \in \langle x \rangle$.

4. Suppose that $x^n = e$. Put $k = o(x)$. Divide k into n, to get $n = qk + r$ where $0 \leq r < k$. Now, we have

$$e = x^n = x^{qk+r} = x^{qk}x^r = (x^k)^q x^r = e^q x^r = x^r,$$

so $x^r = e$. Since k was the least natural number with $x^k = e$, and $r < k$, we must have $r = 0$, so $n = qk$, so $k \mid n$.

5. We wish to know how many distinct values x^n can take. Well, we have

$$\begin{aligned} x^n = x^m &\iff x^{n-m} = e \\ &\iff o(x) \mid n - m \\ &\iff n \equiv m \pmod{o(x)}. \end{aligned}$$

Thus the number of distinct values is the same as the number of congruence classes modulo $o(x)$, which is precisely $o(x)$.

From Qustion 3, we know that $\langle x \rangle$ is a subgroup of G, so by Lagrang's Theorem $|\langle x \rangle| \mid o(G)$, so since $|\langle x \rangle| = o(x)$ we have $o(x) \mid o(G)$.