1. Suppose first that F is one-to-one. [We want to show f is one-to-one.] Let $x, y \in A$ with $f(x) = f(y)$. Put $X = \{x\}$, $Y = \{y\}$. Then $F(X) = \{f(x)\} = \{f(y)\} = F(Y)$, so $F(X) = F(Y)$, so $X = Y$, i.e. $\{x\} = \{y\}$. Since $x \in \{x\}$ we have $x \in \{y\}$, so $x = y$.

Conversely, suppose f is one-to-one, and let $X, Y \in \mathcal{P}(A)$ with $F(X) = F(Y)$. We want to show that $X = Y$. So let $x \in X$. Then $f(x) \in \{\, f(a) : a \in X \,\}$, i.e. $f(x) \in F(X) = F(Y)$, so $f(x) \in F(Y) = \{\, f(a) : a \in Y \,\}$. Thus $f(x) = f(a)$ for some $a \in Y$. Since f is one-to-one, this implies $x = a$, so since $a \in Y$ we have $x \in Y$. Thus $X \subseteq Y$. Conversely, let $y \in Y$. Then

$$f(y) \in F(Y) = F(X) = \{\, f(b) : b \in X \,\}$$

so there is some $b \in X$ with $f(y) = f(b)$, so $y = b \in X$, so $y \in X$. Hence $Y \subseteq X$, so $X = Y$ as required.

2. We must show f^{-1} is one-to-one and onto.

One-to-one: Let $x, y \in B$ with $f^{-1}(x) = f^{-1}(y)$. Then $f(f^{-1}(x)) = f(f^{-1}(y))$, i.e. $x = y$, as required.

Onto: Let $a \in A$. [We must find some $x \in B$ with $f^{-1}(x) = a$. The only element of B that we know of is $f(a)$, so let us hope that it works...] Put $x = f(a)$. Then $f^{-1}(x) = f^{-1}(f(a)) = a$, as required.

3. Suppose first that x is maximal in A. [We want to show that $f(x)$ is maximal in B, i.e. that if $z \in B$ with $f(x) \preceq_B z$ then $f(x) = z$.] Let $z \in B$ with $f(x) \preceq_B z$. Since f is an order-isomorphism, it is onto, so there is some $y \in A$ with $f(y) = z$. So $f(x) \preceq_B f(y)$, so $x \preceq_A y$, so (since x is maximal) $x = y$, so $f(x) = f(y)$, i.e. $f(x) = z$ as required.

Conversely, suppose $f(x)$ is maximal. Let $y \in A$ with $x \preceq_A y$. Then $f(x) \preceq_B f(y)$. By maximality of $f(x)$, $f(x) = f(y)$, so (since f is one-to-one) $x = y$, as required.

4. Suppose, for a contradiction, that A and B were order-isomorphic. Then there would be some order-isomorphism $f : A \to B$. Now, if $x \in A$ then $x = 1 - \frac{1}{n}$ for some $n \in \mathbb{N}$, and $1 - \frac{1}{n} < 1 - \frac{1}{n+1} \in A$, so x is not maximal in A. Thus $f(x)$ is not maximal in B. Now 1 is maximal in B, so there is no $x \in A$ with $f(x) = 1$, so f is not onto, contradicting the assumption that f is an order-isomorphism.