1. (a) Let P_n be the statement "for all $x \in \mathbb{R}$, $f(nx) = nf(x)$".

Base case: P_1 is true because for any $x \in \mathbb{R}$ we have $f(1x) = f(x) = 1f(x)$.

Inductive step: Let $n \in \mathbb{N}$, and suppose P_n is true. Let $x \in \mathbb{R}$. Then

$$\begin{aligned} f((n+1)x) &= f(nx + x) \\ &= f(nx) + f(x) && \text{(by } (*)\text{)} \\ &= nf(x) + f(x) && \text{(by ind. hyp.)} \\ &= (n+1)f(x). \end{aligned}$$

Hence P_{n+1} is true.

Hence, by induction, P_n is true for all n.

(b) By $(*)$ we have $f(0+0) = f(0) + f(0)$, i.e. $f(0) = f(0) + f(0)$. Subtracting $f(0)$ from both sides we get $0 = f(0)$.

(c) By $(*)$ we have $f(x + (-x)) = f(x) + f(-x)$, so $f(x) + f(-x) = f(0) = 0$. Subtracting $f(x)$ from both sides gives $f(-x) = -f(x)$.

(d) Let $n \in \mathbb{Z}$, $x \in \mathbb{R}$. If $n > 0$ then $f(nx) = nf(x)$ by (a). If $n = 0$ then $f(nx) = f(0) = 0 = nf(x)$ by (b). If $n < 0$ then $-n \in \mathbb{N}$, so $f((-n)x) = (-n)f(x)$ by (a). So, by (c), $f(nx) = -f(-(nx)) = -f((-n)x) = -(-n)f(x) = nf(x)$.

(e) Let $x \in \mathbb{R}$, $n \in \mathbb{N}$. Put $y = \frac{x}{n}$. Then, by (d), $f(ny) = nf(y)$, i.e. $f(n\frac{x}{n}) = nf(\frac{x}{n})$, so $f(x) = nf(\frac{x}{n})$. Dividing by n gives $\frac{1}{n}f(x) = f(\frac{x}{n})$.

(f) Let $q \in \mathbb{Q}$ and $x \in \mathbb{R}$. Then $q = \frac{m}{n}$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Applying (e) we have $f(\frac{x}{n}) = \frac{1}{n}f(x)$, so

$$\begin{aligned} f(qx) &= f(\frac{m}{n}x) \\ &= f(m(\frac{x}{n})) \\ &= mf(\frac{x}{n}) && \text{(by (d))} \\ &= m\frac{1}{n}f(x), && \text{(by (e))} \end{aligned}$$

so $f(qx) = qf(x)$ as required.

2. (a) When we start with $n = 63$, we get the sequence 63, 64, 6, 7, 8, 3, 4, 2, 1. When we start with $n = 60$ we get 60, 61, 62, 63, 64, 6, 7, 8, 3, 4, 2, 1.

(b) If we start with $n = 33$ we get 33, 34, 35, …, until we reach 60, 61, 62, and so on and are into the sequence above.

(c) Let P_n be the statement that the algorithm terminates if we start with the value n.

Base case: if we start with $n = 1$ the algorithm terminates immediately.

Inductive step: Let $n \in \mathbb{N}$ and suppose P_j is true for all $1 \leq j \leq n$. By the hint, we know there is some k with $2^k \leq n+1 < 2^{k+1}$. If $2^k = n+1$, then the sequence will begin $n+1$, k, ... Now, $k < 2^k$ so $k < n+1$ so by inductive hypothesis P_k is true. So, if we had started with k, the sequence would terminate. So, since we have reached a value of k we know that the sequence will terminate.

Suppose instead that we have $2^k < n+1 < 2^{k+1}$. Then the sequence will begin $n+1$, $n+2$, ..., $2^{k+1}-1$, 2^{k+1}, $k+1$, ... If $k+1 \leq n$ then we know that P_{k+1} is true (by ind. hyp.) so as before we know the sequence terminates. Now, $k < 2^k < n+1$, so $k+1 \leq 2^k \leq n$, since for any $m, n \in \mathbb{N}$ we have $m < n \iff M = 1 \leq n$. So $k+1 \leq n$, as required.

So, either way, the sequence terminates when we start with $n+1$, i.e. P_{n+1} is true.

Hence, by induction, P_n is true for all $n \in \mathbb{N}$.

3. (a) ρ is not reflexive because $\emptyset \in A$ and $\emptyset \not\rho\, \emptyset$.

(b) ρ is symmetric: let $x, y \in A$ with $x \,\rho\, y$. Then $x \cap y \neq \emptyset$, so $y \cap x \neq \emptyset$, so $y \,\rho\, x$.

(c) ρ is not antisymmetric. Choose $a, b \in A$ with $a \neq b$. Then $\{a\} \in A$ and $\{a, b\} \in A$ and $\{a\} \,\rho\, \{a, b\}$ and $\{a, b\} \,\rho\, \{a\}$ but $\{a\} \neq \{a, b\}$.

(d) ρ is not transitive. Let a and b be as above. Then $\{a\} \,\rho\, \{a, b\}$ and $\{a, b\} \,\rho\, \{b\}$ but $\{a\} \not\rho\, \{b\}$.