1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for all $x, y \in \mathbb{R}$ we have

$$f(x+y) = f(x) + f(y). \qquad (*)$$

We will show that $f(qx) = qf(x)$ for every $q \in \mathbb{Q}$ and $x \in \mathbb{R}$.

(a) Use induction on n to show that for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have $f(nx) = nf(x)$.

(b) Use the fact that $0 + 0 = 0$ and $(*)$ to show that $f(0) = 0$.

(c) Use the fact that $x + (-x) = 0$ and $(*)$ to show that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(d) Put these together to deduce that $f(nx) = nf(x)$ for all $n \in \mathbb{Z}$, $x \in \mathbb{R}$.

(e) Use the fact that this holds for $y = \frac{x}{n}$ to show that $f(\frac{x}{n}) = \frac{1}{n}f(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.

(f) Deduce that if $q \in \mathbb{Q}$ and $x \in \mathbb{R}$ then $f(qx) = qf(x)$.

2. Consider the following algorithm. We start with some number n. If $n = 1$, we stop. If $n = 2^k$ for some k, we replace our value of n with k. Otherwise we replace it with $n + 1$. For example, if we start with $n = 6$ we get the values 6, 7, 8, 3, 4, 2, 1. If we start with $n = 28$ we get 28, 29, 30, 31, 32, 5, 6, 7, 8, 3, 4, 2, 1.

(a) Find the sequence of numbers we get when we start with $n = 63$, and the sequence we get when we start with $n = 60$.

(b) What do we get if we start with $n = 33$? (Do not write out the sequence of values in full).

(c) For every $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ with $2^k \leq n < 2^{k+1}$. Since $k < 2^k$ we have $k < n$. Use this fact to prove by complete induction that the sequence terminates for any starting value of $n \in \mathbb{N}$.

3. Let S be a set with at least 2 elements, and let $A = \mathcal{P}(S)$. Define a relation ρ on A by declaring that, for $x, y \in A$, $x \mathrel{\rho} y$ iff $x \cap y \neq \emptyset$. Determine whether or not ρ is (a) reflexive, (b) symmetric, (c) antisymmetric, and (d) transitive.