NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. Use the Euclidean Algorithm for $\mathbb{R}[x]$ to find a greatest common divisor in $\mathbb{R}[x]$ of $a(x) = x^3 + x^2 - 2x - 8$ and $b(x) = x^2 - 4x + 4$.

2. Let $a(x), b(x) \in \mathbb{R}[x]$ with $a(x), b(x) \neq 0$. We will show that there exists a greatest common divisor $d(x)$ of $a(x)$ and $b(x)$, and $u(x), v(x) \in \mathbb{R}[x]$ with $d(x) = a(x)u(x) + b(x)v(x)$. Our proof uses a similar idea to the proof of Theorem 16 in the Lecture Notes for Week 6.

Put $S = \{\, c(x) \in \mathbb{R}[x] : (\exists u(x), v(x) \in \mathbb{R}[x])(c(x) = a(x)u(x)+b(x)v(x)) \,\}$. Put $T = \{\, \deg c(x) : c(x) \in S \,\}$.

(a) Show that $0 \in S$, so $-\infty \in T$.

(b) Let $T' = T \setminus \{-\infty\}$. Show that T' has a least element, n say.

(c) Since $n \in T'$, there is some $d(x) \in S$ with $\deg d(x) = n$. Thus there exist $u_d(x), v_d(x)$ with $d(x) = a(x)u_d(x) + b(x)v_d(x)$. Use the Division Algorithm to divide $d(x)$ into $a(x)$, and rearrange your equations to write the remainder as a linear combination of $a(x)$ and $b(x)$. [In other words, show that the remainder is in S.]

(d) Deduce that the remainder in the previous part must be 0, in other words that $d(x) \mid a(x)$.

(e) Use a similar argument to show that $d(x) \mid b(x)$.

(f) Suppose $c(x)$ is a common divisor of $a(x)$ and $b(x)$. Show that $c(x) \mid d(x)$.

3. Let $G = \{\alpha, \beta, \gamma, \delta, \varepsilon\}$. Given that $*$ is a group operation on G, we will complete the following Cayley Table for $*$:

$*$	α	β	γ	δ	ε
α		β			
β		δ		γ	
γ					
δ					
ε					

(a) What is the identity element for $*$? This lets us fill in the row and column for that element.

(b) The "once per row and once per column" rule (which from now on we will refer to as 1R1CR) allows two possibilities for $\beta * \varepsilon$. What are they, and which must be the true value for $\beta * \varepsilon$?

(c) Any element of a group commutes with its inverse (so for all $x \in G$, if $x * y = e$ then $y * x = e$). This lets us fill in one more entry since we have found one pair with $x * y = e$. Which pair is it?

(d) Again the 1R1CR allows two possibilities for the two entries $\gamma * \beta$ and $\delta * \beta$. What are these possibilities, and which is which?

(e) We know that $\beta * \beta = \delta$. This lets us find $\delta * \delta$, as

$$\delta * \delta = (\beta * \beta) * (\beta * \beta) = \beta * ((\beta * \beta) * \beta),$$

and by now we know $\beta * x$ and $x * \beta$ for all x. What is $\delta * \delta$?

(f) Use the 1R1CR to find the remaining entries and complete the table.