NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. Define a relation $\preceq$ on $\mathbb{N}$ by declaring that for $x, y \in \mathbb{N}$,

$$x \preceq y \iff x = y \vee x^2 \leq y.$$

Show that $\preceq$ is a partial order on $\mathbb{N}$, but not a total order.

2. Let $A = \{1, 2, 3 \ldots, 20\}$ and $B = \{\, n \in \mathbb{N} : n \mid 20 \,\}$.

(a) Draw lattice diagrams for $(A, |)$ and $(B, |)$.

(b) Find the least upper bound for $\{1, 2, 5\}$ in B.

(c) Find a subset of A which has no least upper bound.

3. Let $(A, \preceq)$ be a poset with the least upper bound property. Let $S \subseteq A$ with $S \neq \emptyset$. Suppose S has at least one lower bound. Put $L_S = \{\, b \in A : b \text{ is a lower bound for } S \,\}$. Show that L_S is nonempty and has at least one upper bound.

From the least upper bound property, we know that $\sup L_S$ exists. Put $g = \sup L_S$. Show that g is a *greatest lower bound* for S, in other words that

- g is a lower bound for S, and
- if b is a lower bound for S then $b \preceq g$.

4. Define a relation ρ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ by declaring that, for $(x, y), (u, v) \in \mathbb{R}^2$,

$$(x, y)\ \rho\ (u, v) \iff x^2 + y^2 = u^2 + v^2.$$

Show that ρ is an equivalence relation. What is the equivalence class $T_{(x,y)}$ of the element (x, y)?