Department of Mathematics

MATHS 255 Assignment 11

1. Let (s_n) be a sequence in $\mathbb{R}$.

(a) Show that if (s_n) converges then (s_n) is a Cauchy sequence.

(b) Show that if (s_n) is a Cauchy sequence then (s_n) is bounded.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that $f + g$, fg and $f \circ g$ are all continuous.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a, b \in \mathbb{R}$ with $a, b \neq 0$. Define a new function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = af(bx)$. Prove from first principles that g is continuous. [“From first principles” here means that you should give a proof using ε and δ rather than quoting results such as those in the previous question.]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that the following are equivalent:

(1) f is continuous.

(2) for every open set U, $f^{-1}(U)$ is open.

(3) for every closed set C, $f^{-1}(C)$ is closed.

(4) for every sequence (s_n) in $\mathbb{R}$ and every $a \in \mathbb{R}$, if $s_n \to a$ as $n \to \infty$ then $f(s_n) \to f(a)$ as $n \to \infty$.

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Show that fg is differentiable, with $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ for all $x \in \mathbb{R}$.