NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date**. Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet: these are available from outside the Resource Centre.

1. Let F be a field, and let $a, b \in F$ with $b \neq O_F$. Show that the equation $a = bx$ has a unique solution $x \in F$.

2. Let F be an ordered field. Let $\leq$ be the relation defined by $x \leq y \iff (y - x \in P \vee x = y)$. Prove that for all $a, b, c \in F$ the following hold.

(a) If $a < b$ then $a + c < b + c$.

(b) If $a \leq b$ then $a + c \leq b + c$.

(c) If $a < b$ and $0_F < c$ then $ac < bc$.

(d) If $a \in P$ then $\frac{1}{a} \in P$. [Hint: use contradiction: consider the other possibilities for $\frac{1}{a}$.]

(e) If $0_F < a < b$ then $0_F < \frac{1}{b} < \frac{1}{a}$. [Hint: multiply $(b - a)$ by $\frac{1}{a} \cdot \frac{1}{b}$.]

3. For each $n \in \mathbb{N}$, let $a_n, b_n \in \mathbb{R}$ with $a_n < b_n$. Suppose that for each n, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. The goal of this question is to prove that $\bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset$.

(a) Let $a, b, c, d \in \mathbb{R}$ with $c \leq d$. Prove that $[c, d] \leq [a, b]$ if and only if $a \leq c$ and $d \leq b$.

(b) Let $c \in \mathbb{R}$. Prove that $c \in \bigcap_{n \in \mathbb{N}} [a_n, b_n]$ if and only if c is both an upper bound for $\{ a_n : n \in \mathbb{N} \}$ and a lower bound for $\{ b_n : n \in \mathbb{N} \}$.

(c) Use induction to prove that for all $n, k \in \mathbb{N}$, $a_n \leq a_{n+k}$ and $b_{n+k} \leq b_n$.

(d) Deduce that for all $m, n \in \mathbb{N}$, $a_m \leq b_n$.

(e) Show that $\{ a_n : n \in \mathbb{N}\}$ is non-empty and bounded above, so it has a least upper bound. Put $c = \sup\{ a_n : n \in \mathbb{N}\}$.

(f) Show that c is a lower bound for $\{ b_n : n \in \mathbb{N} \}$.

(g) Deduce that $\bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset$.

4. Let (s_n) be a sequence in $\mathbb{R}$, and let $L \in \mathbb{R}$. Define new sequences (a_n) and (b_n) by declaring that, for all $n \in \mathbb{N}$, $a_n = s_{2n-1}$ and $b_n = s_{2n}$. Prove that $s_n \to L$ if and only if $a_n \to L$ and $b_n \to L$.