1. (10 marks)

(a) We want to show that for all $M > 0$, there exists $N = N(M) > 0$, such that for all $x > N$, we have $f(g(x)) > M$. Let $M > 0$ be given. Since $\lim_{y\to\infty} f(y) = \infty$, there exists $K > 0$ such that for all $y > K$, $f(y) > M$.

Since $\lim_{x\to\infty} g(x) = \infty$, there exists $H > 0$ such that for all $x > H$, $g(x) > K$.

Let $x > H$ be given. Since $g(x) > K$, $f(g(x)) > M$. Therefore we choose N to be H. For all $x > N$, we have $f(g(x)) > M$.

(b) We want to show that for all $M > 0$, there exists $N = N(M) > 0$, such that for all $x > N$, we have $(f + g)(x) > M$. Since $\lim_{x\to\infty} f(x) = \infty$, there exists $K_1 > 0$ such that for all $x > K_1$, $f(x) > M$. Since $\lim_{x\to\infty} g(x) = \infty$, there exists $K_2 > 0$ such that for all $x > K_2$, $g(x) > M$. Let $N = \max(K_1, K_2) > 0$. For all $x > N$, $f(x) + g(x) > M + M > M$.

(c) We want to show that for all $M > 0$, there exists $N = N(M) > 0$, such that for all $x > N$, we have $kf(x) > M$. Let $M > 0$ be given. Since $\lim_{y\to\infty} f(y) = \infty$, if $M_1 > 0$ is chosen, there exists $H > 0$ such that for all $y > H$, $f(y) > M_1$. Suppose $x > H$.

$$kf(x) > kM_1 = M$$

if we choose M_1 so this happens, i.e. $M_1 = M/k$. Taking N to be H, for all $x > N$, we have $kf(x) > M$.

(d) We want to show that for all $M < 0$, there exists $N = N(M) > 0$, such that for all $x > N$, we have $-f(x) < M$. Let $M < 0$ be given. But since $\lim_{x\to\infty} f(x) = \infty$, there exists $K > 0$ such that for all $x > K$, $f(x) > -M$. For all $x > K$, $-f(x) < M$. Taking N to be K, for all $x > N$, we have $-f(x) < M$.

(e) We want to show that for all $M > 0$, there exists $N = N(M) < 0$, such that for all $x < N$, we have $f(-x) > M$. Let $M > 0$ be given. But since $\lim_{x\to\infty} f(x) = \infty$, there exists $K > 0$ such that for all $x > K$, $f(x) > M$. Take N to be $-K$, for all $x < N$, we have $f(-x) > M$.