1. (a) $\Rightarrow$ (3 marks) Let x and y in G be given. Suppose $x \sim y$. Then there exists $a \in G$ such that x and y are in aH. Thus there exist $h \in H$ and $k \in H$ such that $x = ah$ and $y = ak$. We claim $x^{-1}y \in H$. But $x^{-1}y = (ah)^{-1}(ak) = (h^{-1}a^{-1})(ak) = ((h^{-1}a^{-1})a)k = (h^{-1}(a^{-1}a))k = (h^{-1}e)k = h^{-1}k \in H$.

$\Leftarrow$ (3 marks) Suppose x and y in G are given, and $x^{-1}y \in H$. Let $h = x^{-1}y$. Then $y = xh$. Also we have $x = xe \in xH$. Thus x and y are in the same element, xH, of Ω.

(b) (3 marks) We claim the subgroup $H = n\mathbb{Z} = \{nm : m \in \mathbb{Z}\}$. H_1 is a subgroup since it contains e, and is closed under the group operation, and taking inverses. We have x congruent to y mod n iff $n|x - y$ iff there exists $m \in \mathbb{Z}$ such that $x - y = mn$ iff $x - y \in H$. Thus congruence mod n is the equivalence relation of (a) where $G = \mathbb{Z}$ and $H = n\mathbb{Z}$.

2. (a) (6 marks) We use Prop 1.32. H_1 is a subgroup since it contains e, and is closed under the group operation, and taking inverses. Similarly H_2 is a subgroup since it contains e, and is closed under the group operation, and taking inverses, because $\alpha^2 = e$. Similarly H_3 is a subgroup because $\beta^2 = e$. Similarly H_4 is a subgroup because $\gamma^2 = e$. H_5 is not a subgroup since $\phi^2 = \psi \notin H_5$, and H_5 is not closed under the group operation. Similarly H_6 is not a subgroup since $\psi^2 = \phi \notin H_6$. H_7 is a subgroup, because $\phi^2 = \psi, \psi^2 = \phi, \phi\psi = e$, and $\psi\phi = e$, giving H_7 is a subgroup since it contains e, and is closed under the group operation, and taking inverses. H_8 is not a subgroup because $\alpha\beta = \phi \notin H_8$. H_9 is not a subgroup because $|H_9| \not| \, |S_3|$, whereas by Lagrange's Th the number of elements in any subgroup is a divisor of 6.

(b) (4 marks) There are three cosets. $H_2 = \{e, \alpha\}$. $\beta H_2 = \{\beta e, \beta\alpha\} = \{\beta, \phi\} = \phi H_2$. $\gamma H_2 = \{\gamma e, \gamma\alpha\} = \{\gamma, \psi\} = \psi H_2$.

$H_7 = \{e, \phi, \psi\}$. The other left coset is $\{\alpha, \beta, \gamma\}$, because it contains three elements of S_3, hence all of the elements not in H_7.