1. (7 marks) First, we claim that if $f : A \to A$ and $g : A \to A$ are bijections, then so is the composite, $f \circ g$, defined by $f \circ g(x) = f(g(x))$ for all $x \in A$. This holds by Th 5.2.3 of Schumacher. Thus $\circ$ is a binary operation on this set, which we call S_A.

By Th 5.2.5 of Schumacher, $\circ$ is associative.

We show the identity function $e : A \to A$, defined by $e(x) = x$ for all $x \in A$, is an identity for the group S_A. We claim that for all $f \in S_A$,

$$f \circ e = f$$

and

$$e \circ f = f.$$

To prove this, let $f \in S_A$ be given, and let $x \in A$ be given. Then

$$f \circ e(x) = f(e(x)) = f(x) = e(f(x)) = e \circ f(x).$$

This proves our claim.

We show that for all $f \in S_A$, there is an inverse in the sense of group theory for f, i.e. $y \in S_A$ such that $f \circ y = e = y \circ f$. Of course, we take y to be the inverse f^{-1} of f in the sense of bijective functions, given by definition 5.2.8 of Schumacher. By Th 5.2.9 of Schumacher, we have, for all $x \in A$,

$$f \circ f^{-1}(x) = x = f^{-1} \circ f(x).$$

This gives

$$f \circ f^{-1} = e = f^{-1} \circ f,$$

and shows f^{-1} to be group theoretic inverse of f.

2. (7 marks) Let $x \in G$ and $y \in G$ be given. We want to show $xy = yx$.

We know that for all $z \in G$, $z^2 = e$, or equivalently,

$$z^{-1} = z. \qquad (1)$$

We need to give a Lemma.

Lemma. Let G be a group. For all $x \in G$ and $y \in G$, $(xy)^{-1} = y^{-1}x^{-1}$.

Proof. Let e be the identity and let x and y in G be given.

$$\begin{aligned}
(y^{-1}x^{-1})(xy) &= y^{-1}(x^{-1}(xy))\text{assoc law} \\
&= y^{-1}((x^{-1}x)y)\text{assoc law} \\
&= y^{-1}(ey)\text{def of inverse} \\
&= y^{-1}(y)\text{def of } e \\
&= e.
\end{aligned}$$

Thus $(y^{-1}x^{-1})$ is a left inverse of xy and hence is the inverse. (Or we can say "Similarly, $(xy)(y^{-1}x^{-1}) = e$, and hence $(y^{-1}x^{-1}) = (xy)^{-1}$.) That ends the proof of the Lemma; now we carry on with the main proof.

$$\begin{aligned}(xy) &= (xy)^{-1} \text{ by (1)} \\ &= y^{-1}x^{-1} \text{ by Lemma} \\ &= yx \text{ by (1)}\end{aligned}$$

Hence G is abelian.

3. (7 marks) By Prop 1.52 of class notes on Group Theory, we want to show:

1)The identity e_H of H is in $f(G)$,

2) for all x and y in $f(G)$, $xy \in f(G)$, and

3) for all $x \in f(G)$, $x^{-1} \in f(G)$.

For 1), with e_G the identity of G, by Prop 1.26, $f(e_G) = e_H$. This gives $e_H \in f(G)$.

Foe 2), we let x and y in $f(G)$ be given. Take a and b in G such that $x = f(a)$ and $y = f(b)$. Then $f(ab) = f(a)f(b)$, since f is a homomorphism, and this equals xy, giving $xy \in f(G)$.

For 3) we let $x \in f(G)$ be given. Take $a \in G$ such that $f(a) = x$. We give a lemma.

Lemmma Given $f : G \to H$, a homomorphism of groups, for any $x \in G$, $(f(x))^{-1} = f(x^{-1})$.

Proof. Let $x \in G$ be given.

$$\begin{aligned}f(x^{-1})f(x) &= f(x^{-1}x) \text{ f a homomorphism} \\ &= f(e_G) \\ &= e_H \text{ Prop 1.26 of class notes}\end{aligned}$$

Hence $f(x^{-1}) = (f(x))^{-1}$, ending the proof of the Lemma.

We continue with the main proof.

$$f(a^{-1}) = (f(a))^{-1} = x^{-1},$$

and $x^{-1} \in f(G)$. Since 1),2) and 3) hold, $f(G)$ is a subgroup of H.