NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date.** Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet. These are available from outside the Resource Centre. PLEASE SHOW ALL WORKING. Also if we believe you have COPIED someone else's script or that you have let someone else COPY YOUR SCRIPT, then you will get NO MARKS.

1. Prove or give a counterexample: If $f : X \to Y$ is a function between the sets X and Y and B, C are subsets of Y then $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$.
soln: It is true as given:

$$\begin{aligned} a \in f^{-1}(B \cup C) &\Leftrightarrow f(a) \in B \cup C \\ &\Leftrightarrow f(a) \in B \text{ or } f(a) \in C \\ &\Leftrightarrow a \in f^{-1}(B) \text{ or } a \in f^{-1}(C) \\ &\Leftrightarrow a \in f^{-1}(B) \cup f^{-1}(C). \end{aligned}$$

[4 marks]

2. Let

$$A = \{x \in \mathbb{R} : x = 1 - \frac{1}{n} \text{ for some } n \in \mathbb{N}\},$$

and let $B = A \cup \{1\}$. Define f to be the function $f : A \to B$ given by $f(x) = \frac{2x-1}{x}$ for $x \neq 0$, and $f(0) = 1$.

(a) Show that f is a bijection. (Hint: First show that the formulae for f, applied to elements of A does yield elements of B as asserted.)
soln: (Note that if $x = 1 - 1/n = (n-1)/n$ $(n \geq 2)$ then

$$\frac{2x-1}{x} = \frac{n-2}{n-1} = 1 - \frac{1}{n-1} \quad * .$$

Also we have $f(0) = 1$ so always f takes values in B.) If $b \in B$ then either $b = 1 = f(0)$ or $b = 1 - 1/m$ for some $m \in \mathbb{N}$. In the latter case

$$b = f(1 - \frac{1}{n}) \quad \text{where} \quad n = m + 1 \geq 2.$$

Thus $f : A \to B$ is shown to be surjective.
Now suppose that $a, b \in A$ are such that $f(a) = f(b)$. If $a = 0$ then $f(b) = 1$. Thus $b = 0$ since for all $n \in \mathbb{N}$, $1 - \frac{1}{n-1} < 1$. So in this case $a = b$. By the same reasoning if $b = 0$ and $f(a) = f(b)$ then $a = 0$. Now suppose $f(a) = f(b)$ and $a \neq 0$ and $b \neq 0$. Then $a = 1 - 1/n$ for some integer $n \geq 2$ and $b = 1 - 1/m$ for some integer $m \geq 2$. Now from $*$ above $f(a) = f(b)$ implies

$$1 - \frac{1}{n-1} = 1 - \frac{1}{m-1}.$$

Thus

$$\frac{1}{n-1} = \frac{1}{m-1} \quad \Rightarrow \quad n = m,$$

and so $a = b$. We have shown that in all case $f(a) = f(b)$ implies $a = b$. Thus f is injective. With surjectivity as established already we have that f is bijective as claimed. [6 marks]

(b) Show that if A and B are considered as posets by restricting the usual ordering on $\mathbb{R}$, then f is not an order isomorphism.
soln: $1 = f(0) > f(1/2) = 0$ and yet $0 \leq 1/2$ thus f is not order preserving as required for an order isomorphism. [2 marks]

3. Prove by induction that, for every $n \in \mathbb{N}$, $11^n - 4^n = 7\ell$ for some $\ell \in \mathbb{N}$.

soln: Let P_n be the proposition (or statement) that $11^n - 4^n$ is divisible by 7 for a given $n \in \mathbb{N}$. Note that if $n = 1$ then $11^n - 4^n = 11 - 4 = 7 = 7 \cdot 1$. Thus P_1 is shown to be true. [2 marks so far]

Suppose for some given $k \in \mathbb{N}$, P_k is true, i.e., $11^k - 4^k = 7m$ for $m \in \mathbb{N}$. Consider now $11^{k+1} - 4^{k+1}$. We have

$$11^{k+1} - 4^{k+1} = 11 \cdot 11^k - 4 \cdot 4^k = 7 \cdot 11^k + 4(11^k - 4^k).$$

Thus, given P_k we have

$$11^{k+1} - 4^{k+1} = 7 \cdot 11^k + 4(7m) = 7(11^k + 4m).$$

Since $11^k + 4m \in \mathbb{N}$ this shows that P_k implies P_{k+1}.

Thus by the Principle of Induction it follows that P_n is true for all $n \in \mathbb{N}$. That is $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$. [5 marks = 2 marks for setting out argument and 3 marks for details.]