1. Suppose that X is a poset with a partial ordering $\leq$. Show that X has at most one least element.
soln: [5 marks]
Suppose x and y are least elements of X. Then $x \leq y$ since $y \in X$ and x is a least element of X (which means that for all $a \in X$, $x \leq a$). Similarly $y \leq x$ since $x \in X$ and y is a least element of X. Thus we have $x \leq y$ and $y \leq x$ which implies that $x = y$ since $\leq$ is antisymmetric.

2. Explain why the collection of sets $A = \{2, 3, 7\}, B = \{4, 5, 6\}, C = \{1, 8, 3\}$ is not a partition of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
soln: [2 marks]
Because $A \cap C = \{3\}$ and so $A \cap C \neq \emptyset$ as required.

3. Define a relation $\sim$ on the Cartesian plane $\mathbb{R}^2$ by,

$$(x, y) \sim (u, v) \text{ if and only if } 3x + 2y = 3u + 2v,$$

for (x, y) and (u, v) elements of $\mathbb{R}^2$.

(a) Show that $\sim$ is an equivalence relation.
soln: [6 marks]
Reflexive: Let $(x, y) \in \mathbb{R}^2$. Then $3x + 2y = 3x + 2y$, so $(x, y) \sim (x, y)$.
Symmetric: Let $(x, y), (u, v) \in \mathbb{R}^2$ with $(x, y) \sim (u, v)$. Then $3x + 2y = 3u + 2v$, so $3u + 2v = 3x + 2y$, that is $(u, v) \sim (x, y)$.
Transitive: Let $(x, y), (u, v), (z, w) \in \mathbb{R}^2$ with $(x, y) \sim (u, v)$ and $(u, v) \sim (z, w)$. Then $3x+2y = 3u + 2v$ and $3u + 2v = 3z + 2w$. So $3x + 2y = 3z + 2w$, that is $(x, y) \sim (z, w)$.

(b) Give a geometrical description of the equivalence class $[(a, b)]$.
soln: [4 marks]
$(x, y) \in [(a, b)] \Leftrightarrow 3x + 2y = 3a + 2b \Leftrightarrow y = -\frac{3}{2}x + \frac{3a+2b}{2}$. So $[(a, b)]$ is the (straight) line through the point $(0, \frac{3a+2b}{2})$ with slope $-3/2$.

4. Prove that if functions $f : A \to B$ and $g : B \to C$ are one-to-one then so is $g \circ f$.

soln: [5 marks]
Suppose for $a_1, a_2 \in A$ we have $g \circ f(a_1) = g \circ f(a_2)$. That is $g(f(a_1)) = g(f(a_2))$. Since g is one-to-one we have

$$g(f(a_1)) = g(f(a_2)) \text{ implies } f(a_1) = f(a_2),$$

so $f(a_1) = f(a_2)$. But f is one-to-one and so

$$f(a_1) = f(a_2) \text{ implies } a_1 = a_2.$$

Thus $a_1 = a_2$. $\square$

5. Suppose that X is a poset with a partial ordering $\leq$ and A a subset of X. Define L_A by,

$$L_A := \{x \in X \ : \ x \text{ is a lower bound for A}\}.$$

Suppose that L_A has a least upper bound b. Show that b is a greatest (or largest) lower bound for A.

soln: [8 marks]

First we will show that b is a lower bound for A. With a view to contradiction suppose not. That is suppose that there is $a \in A$ such that $a < b$. Since $a \in A$ it is clear that for all $x \in L_A$ we have $x \leq a$ (as $x \in L_A$ means x is a lower bound for A). Thus a is an upper bound for L_A which is less than b. This contradicts that b is a least upper bound as assumed. Thus we conclude there are no elements $a \in A$ such that $a < b$. That is for all $a \in A$, $b \leq a$, in other words

$$b \text{ is a lower bound for } A \quad *$$

.

On the other hand we know that b is an upper bound for L_A so if c is any lower bound for A, we have $c \geq b$. This with $*$ means that b is a glb for A. $\square$

Total marks: 30