1. Let A and B be sets.

(a) Prove that $B \setminus (B \setminus A) = A \cap B$.
soln: [5 marks (1 per line or equivalent)]

$$\begin{aligned} x \in B \setminus (B \setminus A) &\Leftrightarrow x \in B \text{ and } x \notin (B \setminus A) \\ &\Leftrightarrow x \in B \text{ and } (x \notin B \text{ or } x \in A) \\ &\Leftrightarrow (x \in B \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \in A) \\ &\Leftrightarrow x \in B \text{ and } x \in A \\ &\Leftrightarrow x \in A \cap B \end{aligned}$$

(b) Prove the statement

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$$

soln: [5 marks]
Suppose $S \in \mathcal{P}(A) \cup \mathcal{P}(B)$. That is $S \in \mathcal{P}(\mathcal{A})$ or $S \in \mathcal{P}(B)$. $S \in \mathcal{P}(A)$ means $S \subseteq A$. Similarly $S \in \mathcal{P}(B) \Leftrightarrow S \subseteq B$. Thus $S \subseteq A$, or $S \subseteq B$. But $A \subseteq A \cup B$ and $B \subseteq A \cup B$ and thus either case from the last line implies $S \subseteq A \cup B$. That is $S \in \mathcal{P}(A \cup B)$. $\square$

(c) Show that it is not necessarily true that

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$$

soln: [6 marks (= 2 for the counterexample, 2 for why it is a counterexample and 2 for evidence that you did the working.]
Let $A = \{1\}$, $B = \{2\}$. Then $\{1,2\} \in \mathcal{P}(A \cup B)$. But $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$. To see the latter observe first that $\{1,2\}$ is not a subset of A. That is $\{1,2\} \notin \mathcal{P}(A)$. Similarly $\{1,2\} \notin \mathcal{P}(B)$. Thus $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$ as claimed. $\square$

2. Indicate whether each of the following relations on the given set is reflexive, symmetric, antisymmetric, or transitive. Explain each answer.

(a) $A = \{s : s \text{ is a student in MATHS255}\}$. $x \rho y$ iff x is no older than y. [8 marks =2+2+2+2]
soln: Reflexive. (A student is the *same* age as him or herself therefore no older).
Not symmetric. (For example, if Jo is younger than Sanja then (Jo , Sanja) $\in \rho$ but (Sanja, Jo) $\notin \rho$).
Not antisymmetric (probably). (There may be two students of the same age.)
Transitive. (Let the age of a student s is given by $A(s)$. Then $x \rho y \Leftrightarrow A(x) \leq A(y)$. But certainly $A(x) \leq A(y)$ and $A(y) \leq A(z)$ implies $A(x) \leq A(z)$.)

(b) $B = \{x : x \in \mathbb{Z} \text{ and } x > 0\}$. $\ell \rho m$ iff $\ell + m = 5$. [8 marks]
soln: Note $\rho = \{(1,4),(4,1),(2,3),(3,2)\}$. (This was not asked for in the question. But it is useful!)

Not reflexive. (By inspection of the set ρ it is clear that there is *not* any $x \in B = \mathbb{N}$ such that $(x, x) \in \rho$. Alternatively you could observe that, for $x \in B = \mathbb{N}$, $x + x = 2x$ is even and so never equal to 5.)

Is symmetric. (If $x\rho y$ is true then $x + y = 5$, thus $y + x = 5$, i.e. $y\rho x$ is true.)

Not antisymmetric. (E.g. $1 \ \rho \ 4$ and $4 \ \rho \ 1$ and yet $1 \neq 4$.)

Not transitive. (E.g. $1 \ \rho \ 4$ and $4 \ \rho \ 1$ and yet $(1, 1) \notin \rho$.)

(c) $C = \{a, b, c\}$ (distinct elements). $\rho = \{(b, b), (b, c), (c, a), (b, a), (a, b)\}$. [8 marks]
soln:

Not reflexive. ($(a, a) \notin \rho$.)

Not symmetric. ($(c, a) \in \rho$ but $(a, c) \notin \rho$.)

Not antisymmetric. ($(b, a) \in \rho$ and $(a, b) \in \rho$ but $a \neq b$.)

Not transitive. ($(a, b) \in \rho$ and $(b, a) \in \rho$ but $(a, a) \notin \rho$.)

(d) $D = \{\ell : \ell$ is a straight line in the Cartesian plane $\}$. $\ell\rho m$ iff ℓ intersects m. [8 marks]
soln:

Is reflexive. (Each straight line shares at least one point with itself, i.e. it intersects itself.)

Is symmetric. (A line ℓ intersects a line m iff there is a point $X = (x, y)$ in the plane such that X is on ℓ and X is on m. Clearly then m intersects ℓ.)

Not antisymmetric. (Distinct lines can intersect. This with symmetry means not antisymmetric.)

Not transitive. (Distinct Parallel lines may intersect a common 3rd line.)

(e) $E = \{p : p$ is a staff member at the University of Auckland$\}$. $p\rho q$ iff p has shaken hands with q. [8 marks]
soln:

Not reflexive. (Otherwise I want to know who goes around shaking their own hands.)

Is symmetric. (If Majid has shaken hands with Bethana then Bethana has shaken hands with Majid!)

Not antisymmetric. (For example, if Majid has shaken hands with Bethana then Bethana has shaken hands with Majid and yet Majid and Bethana are distinct humans.)

Not transitive. (If Majid has shaken hands with Bethana and Bethana has shaken hands with Alan it is not necessarily the case that Majid and Alan have shaken hands.)