1. For each of the sentences in quotations, answer the following questions.

(i) Is it a statement? A predicate?

(ii) Are predicates involved? If so list any free or bound variables?

(iii) Where possible express each symbolically in terms of the statements, predicates, sets and relations given and using the notation of logic as in the lectures.

(iv) Where appropriate decide if it is a tautology, a contradiction or neither of these. Give a reason

(a) "A and B are true or B is false." Here A and B are statements.

soln.: (i) Statement. [1 mark]
(ii) No predicates involved explicitly. [1 mark]
(iii) $(A \wedge B) \vee \sim B$ [1 mark]
(iv) Neither. Reason: If A and B both true then the statement is true. If A is false and B true then the statement is false. [2 marks]

(b) "He is stronger than Helen Clark."

soln.: (i) Predicate. [1 mark]
(ii) 'He' is a free variable. [1 mark]
(iii) Nothing required here. [1 mark] There are various possibilities that could be written. For example we could replace the sentence with the essentially equivalent predicate $S(x)$ meaning "man x is stronger than Helen Clark". Anything like this gets a mark provided nothing incorrect is given.
(iv) Neither. Reason: It is not a statement. (Predicates with free variables have no decidable truth value.) [2 marks]

(c) "If either A is true or B is true then it is not the case that both A and B are true." Here A and B are statements. [5 marks distributed as above]

soln.: (i) Statement.
(ii) No predicates involved explicitly.
(iii) $(A \vee B) \Rightarrow \sim (A \wedge B)$
(iv) Neither. Reason: If A and B are both true then the statement is false. If A and B are both false then the statement is true.

(d) "If A then why not B?". Here A and B are statements. [5 marks distributed as above]

soln.: (i) Neither.
(ii) No predicates involved explicitly.
(iii) Not applicable (NA).
(iv) NA.

(e) If $y + x^2 < 0$ then $y < 0$. [5 marks distributed as above]

soln.: (i) Predicate.
(ii) x and y are free variables .

(iii) $y + x^2 < 0 \Rightarrow y < 0$
(iv) Neither. Reason: It is not a statement. (Predicates with free variables have no decidable truth value).

(f) "Joe never needs help from anybody" in terms of the set of all people 'P', the set of all times 'T', and the predicate $R(x, y, t)$ = 'person x needs help from person y at time t'. (Here Joe is a given particular person.) [5 marks distributed as above]

soln.: (i) Statement.
(ii) Bound variables are the **time** and the **person who might help**. I'd accept y and t.
(iii) Let j represent Joe. $\sim (\exists t \in T, \exists y \in P, R(j, y, t))$. There are other equivalent statements. For example, $\forall t \in T, \forall y \in P, \sim R(j, y, t)$, or $\sim (\exists y \in P, \exists t \in T, R(j, y, t))$ or $\forall y \in P, \forall t \in T, \sim R(j, y, t)$
(iv) Neither. Reason: The statement is not a compound statement.

(g) "Always everybody needs help from somebody" using the same notation as part f. [5 marks distributed as above]

soln.: (i) Statement.
(ii) Bound variables are the **time** and the **person who might help** and the **person who might need help**. I'd accept x, y and t.
(iii) For example $\forall t \in T, \forall x \in P, \exists y \in P, R(x, y, t))$. Any other logically equivalent statements is acceptable. WARNING: $\forall t \in T, \exists y \in P, \forall x \in P, R(x, y, t))$ is not equivalent and is wrong.
(iv) Neither. Reason: The statement is not a compound statement.

2. Assume A, B, and C are statements. Construct truth tables for each of the following pairs of compound statements. State (giving a reason) whether either implies the other or whether they are equivalent.

(a) $A \wedge (B \vee C)$, $(A \vee B) \wedge (A \vee C)$ **soln.:**

A	B	C	$A \wedge (B \vee C)$	$(A \vee B) \wedge (A \vee C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(Students should show their working by having more columns for $B \vee C$ etcetera.) [3 marks] The first implies the second as whenever $A \wedge (B \vee C)$ is true then $(A \vee B) \wedge (A \vee C)$ is true. But they are not equivalent since the converse does hold. [2 marks]

(b) $\sim (A \vee \sim B)$ $\sim A \wedge B$
soln.:

A	B	$\sim(A \vee \sim B)$	$\sim A \wedge B$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	F	F

(Students should show working by having more columns.) [3 marks] The two statements are equivalent as they have the same truth values. [2 marks]

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = 3x^2 + 2$.

(a) Use a carefully expressed direct proof to show that if $f(n) < f(k)$ and $0 < k + n$ then $n < k$.

soln.: Suppose that n, k is a pair of integers such that $f(n) < f(k)$ and $0 < k + n$. [1 mark]
That is $3n^2 + 2 < 3k^2 + 2$. This is equivalent to $3n^2 + 3 - (3k^2 + 2) < 0$. That is $3(n^2 - k^2) < 0$, or equivalently, $n^2 - k^2 < 0$. [1 mark]
Now $n^2 - k^2 = (n + k)(n - k)$ therefore the last inequality is the same as $(n + k)(n - k) < 0$. [1 mark]
Since $k + n > 0$ it follows that

$$n - k < 0$$

and so $n < k$. [2 marks] □

(b) Prove that the statement $(f(x) = f(y) \Rightarrow x = y)$ is false. Explain briefly how you will do this.

soln.: (NB: In the context what this really means is that we should show that $\forall x \forall y (f(x) = f(y) \Rightarrow x = y)$ is false. It suffices therefore to show $\exists x \exists y (f(x) = f(y) \wedge x \neq y)$ is true. That is find a pair of integers x, y such that $f(x) = f(y)$ is true but $x = y$ is false. Such a pair is a a **counterexample** to the assertion that $(f(x) = f(y) \Rightarrow x = y)$ is true for all pairs integers x, y.) Marking starts here:

It is sufficient to find a counterexample. [1 mark]
$x = 1$ and $y = -1$ are integers such that $x^2 = 1 = y^2$ and so $f(x) = 5 = f(y)$ and yet $x \neq y$. [2 marks]
Thus the statement is proved false as required. [1 mark]