THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2002
Campus: City

MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: Answer ALL the questions. The marks add up to 180, 1 per minute.

1. (18 marks)

(a) Prove that if x is an odd integer then x^2 is an odd integer.

(b) Consider the statement:

if x is an even integer then x^2 is an even integer.

Write down the statement converse to this statement and then prove this converse statement is true.

(c) Suppose A and B are sets. Prove that

$$B \setminus (B \setminus A) = A \cap B.$$

2. (18 marks) Let A be a non-empty set and let B be a fixed subset of A. Define a relation $\sim$ on $\mathcal{P}(A)$ by

For $C, D \in \mathcal{P}(A)$ $\quad C \sim D$ if and only if $C \cap B = D \cap B$.

(a) Show that $\sim$ is an equivalence relation on $\mathcal{P}(A)$.

(b) For the particular case where $A = \{1, 2, 3, 4, 5\}$, and $B = \{1, 2, 5\}$, find the equivalence class of $C = \{2, 4, 5\}$, under $\sim$.

3. (16 marks) Give a carefully presented proof by induction that for all $n \in \mathbb{N}$, 3 divides $2^{2n} - 1$.

4. (20 marks)

(a) Show the equation $\overline{7} = \overline{6} \cdot_{12} \overline{x}$ has no solutions in $\mathbb{Z}_{12}$.

(b) Let $a, b, n \in \mathbb{N}$. Suppose there exists an integer c such that $ac \equiv 1 \pmod{n}$. Show that the equation $\overline{a} \cdot_n \overline{x} = \overline{b}$ has a solution $\overline{x} \in \mathbb{Z}_n$, and show that this solution is unique.

5. (8 marks) Let G, H and J be groups, and let $f : G \to H$ and $g : H \to J$ be homomorphisms. Show that $g \circ f$ is a homomorphism.

6. (10 marks) Let G be a group, and let H and K be subgroups. Show that their intersection, $H \cap K$, is a subgroup.

7. (14 marks) Let G be a group with identity e_G. Let H be a group with identity e_H. Let $f : G \to H$ be a homomorphism. Prove that if $\{x \in G : f(x) = e_H\} = \{e_G\}$, then f is one to one, adding words to this calculation.

$$f(x) = f(y). \quad f(x^{-1}y) = f(x^{-1})f(y) = f(x)^{-1}f(y) = f(y)^{-1}f(y) = e_H. \quad x^{-1}y = e_G. \quad x = y$$

Prove the converse.

8. (12 marks) Let A be a nonempty set and let $a \in A$ be given. Let S_A be the group of bijections $f : A \to A$, under composition. Show that $H = \{f \in S_A : f(a) = a\}$ is a subgroup of S_A.

9. (12 marks) Prove that $(0, 1)$ has no least element.

10. (12 marks) Let A and B be subsets of $\mathbb{R}$. Suppose A is nonempty, $A \subset B$, and B is bounded above. Show that the least upper bounds of A and B exist, and satisfy $\operatorname{lub} A \leq \operatorname{lub} B$.

11. (12 marks) Suppose $g : \mathbb{R} \to \mathbb{R}$ satisfies $\lim_{x\to\infty} g(x) = \infty$. Show from first principles that $\lim_{x\to\infty} -0.5g(x) = -\infty$.

12. (18 marks) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. Let a and M be real numbers. Suppose $\lim_{x\to a} f(x) = 0$, and there exists $\delta_1 > 0$ such that for all $x \in (a - \delta_1, a + \delta_1)$, $|g(x)| \leq M$. Show from first principles that $\lim_{x\to a} f(x)g(x) = 0$.

13. (10 marks) Suppose $\{a_n\}$ is a sequence of real numbers converging to 0 as $n \to \infty$. Suppose x is a real number, and for all n, $x \leq a_n$. Show that $x \leq 0$.