MATHS 255: Principles of Mathematics
CLASS TEST

3-4 pm THURSDAY 13th September 2001
Solutions

1. Let $A(n)$ be the predicate:

If n is not prime, then $\exists p \in \mathbf{N}$ p is prime and $p \mid n$.

(a) [5] Write down the negation of $A(n)$,

n is not prime and $\forall p \in \mathbf{N}$, p is not prime or p does not divide n.

(b) [10] Determine with reasons whether or not it is true that $\exists n \in \mathbf{N}$ $A(n)$.

True. Take for example $n = 2$ (any prime would work as well). The hypothesis "n is not prime" is not satisfied, so the conditional statement $A(2)$ is true.

(c) [10] Determine with reasons whether or not it is true that $\forall n \in \mathbf{N}$ $A(n)$.

False. Take for example $n = 1$. 1 is not prime and moreover it has no prime divisor. So $A(1)$ is false.

2. A relation $\lhd$ is defined on $\mathbf{R} \times \mathbf{R}$ by $(a,b) \lhd (c,d)$ if and only if

either $a < c$ or $(a = c \text{ and } b \le d)$.

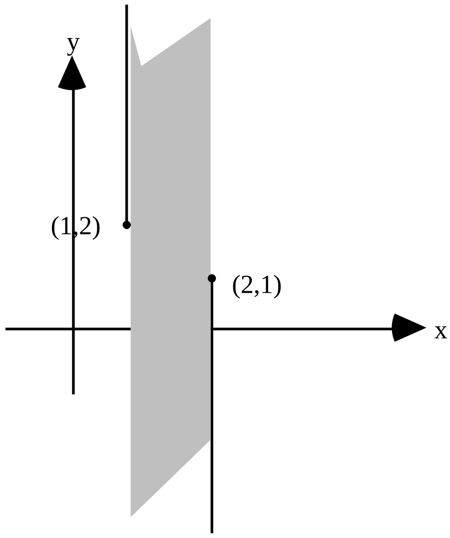
($\lhd$ is sometimes called "dictionary ordering," and in fact it is a total ordering of $\mathbf{R} \times \mathbf{R}$.)

(a) [5] Put the following three elements of $\mathbf{R} \times \mathbf{R}$ in order:

$$\left(\pi,\ \sqrt{10}\right), \left(\frac{22}{7},\ \pi\right), \left(\pi,\ \frac{22}{7}\right) \quad \text{[Hint: } \pi < \frac{22}{7} < \sqrt{10}\text{]}.$$

Ans: $\left(\pi,\ \frac{22}{7}\right) \lhd \left(\pi,\ \sqrt{10}\right) \lhd \left(\frac{22}{7},\ \pi\right)$.

(b) [10] Assuming that $\lhd$ is a total ordering of $\mathbf{R} \times \mathbf{R}$, sketch the "closed interval" $\{(a,b) : (1,2) \lhd (a,b) \lhd (2,1)\}$.



(c) [15] Assuming that $\leq$ is a total ordering on $\mathbf{R}$, but without assuming that $\lhd$ is a total ordering on $\mathbf{R}\times\mathbf{R}$, prove that $\lhd$ is antisymmetric.

Notice first that if $(a,b)\lhd(c,d)$, then $a\leq c$. Now suppose $(a,b)\lhd(c,d)$ and $(c,d)\lhd(a,b)$. Then $a\leq c$ and $c\leq a$, so $a=c$. Since $(a,b)\lhd(c,d)$, we now have that $b\leq d$, and since $(c,d)\lhd(a,b)$, we have that $d\leq b$, So $b=d$. Hence $(a,b)=(c,d)$.

3. [20] Find and prove a formula for $\frac{1}{n}+\sum_{i=2}^{n}\frac{1}{i(i-1)}$ valid for all integers $n\geq 2$,

By testing a few values, we expect that $\frac{1}{n}+\sum_{i=2}^{n}\frac{1}{i(i-1)}=1$ for all $n\geq 2$. We prove it by induction: Let P_n be the statement: $\frac{1}{n}+\sum_{i=2}^{n}\frac{1}{i(i-1)}=1$.

P_2 is true. Proof: $\frac{1}{2}+\frac{1}{2\cdot 1}=1$.

For $k\geq 2$, $P_k\Rightarrow P_{k+1}$ is true. Proof:

$$P_k\Rightarrow\frac{1}{k}+\sum_{i=2}^{k}\frac{1}{i(i-1)}=1$$
$$\Rightarrow\sum_{i=2}^{k}\frac{1}{i(i-1)}=1-\frac{1}{k}$$
$$\Rightarrow\frac{1}{k+1}+\sum_{i=2}^{k+1}\frac{1}{i(i-1)}=\frac{1}{k+1}+\left(1-\frac{1}{k}\right)+\frac{1}{(k+1)k}$$
$$\Rightarrow\frac{1}{k+1}+\sum_{i=2}^{k+1}\frac{1}{i(i-1)}=1$$
$$\Rightarrow P_{k+1}.$$

Hence, by PMI. P_n is true for all $n\geq 2$.

4. [25] Find a greatest common divisor $d(x)$ of the polynomials $a(x) = x^3 + x^2 + x + \bar{1}$, $b(x) = x^4 - x^3 + x + \bar{1}$ in $\mathbf{Z}_2[x]$, and find $u(x),\ v(x) \in \mathbf{Z}_2[x]$ such that $d(x) = a(x)u(x) + b(x)v(x)$.

We use the Euclidean algorithm. Dividing $b(x)$ by $a(x)$, we get quotient $q_1(x) = x$ and remainder $r_1(x) = x^2 + \bar{1}$.
Dividing $a(x)$ by $r_1(x)$, we get quotient $q_2(x) = x + \bar{1}$ and remainder $r_2(x) = \bar{0}$.
Hence, the greatest common divisor (the last non-zero remainder) is $d(x) = r_1(x) = x^2 + \bar{1}$, and since $d(x) = b(x) - a(x)q_1(x)$, we take $u(x) = -q_1(x) = -x = x$ and $v(x) = \bar{1}$, so that $d(x) = a(x)u(x) + b(x)v(x)$.