DEPARTMENT OF MATHEMATICS

MATHS 255FC	Assignment 8 solutions	Due 26th September 2001

1. Prove each of the following from the axioms given in the handout on real numbers.

(a) Given $a, b \in \mathbf{R}$, show there is a unique x such that $a + x = b$.

Firstly $\exists y{:}\, a + y = y + a = 0$ (F5). Also y is unique as $y = 0 + y = (y' + a) + y = y' + (a + y) = y' + 0 = y'$.
Now let $x = b + y$ then $a + x = a + (b + y) = a + (y + b) = (a + y) + b = 0 + b = b$.
Also b is unique as $a + x' = b = a + x \Rightarrow y + a + x' = y + a + x \Rightarrow 0 + x' = 0 + x \Rightarrow x' = x$

(b) If $x = b - a$ is defined to be this x, and $-a$ is defined to be $0 - a$ show:
(i) $b - a = b + (-a)$ (ii) $a(b - c) = ab - ac$ (iii) $0.a = a.0 = 0$ (iv) ab = ac, $a \neq 0 \Rightarrow$ b = c

(i) By above $b - a = x$, $0 - a = y$ and $b - a = x = b + y = b + (0 - a) = b + (-a)$
(iii) $ac = a(0 + c) = a.0 + ac$, so $a.0 = ac - ac = 0$.
(ii) $a(b - c) + ac = a(b + (-c)) + ac = ab + a(-c) + ac = ab + a(c + (-c)) = ab + a.0 = ab$
So $a(b - c) = ab - ac$
(iv) $a \neq 0 \Rightarrow \exists y{:}\, a.y = y.a = 1 \quad a.b = a.c \Rightarrow y.a.b = y.a.c \Rightarrow 1.b = 1.c \Rightarrow b = c$

(c) (i) $x < y \Rightarrow x + z < y + z$ (ii) $1 > 0$ (iii) $x < y \Rightarrow -x > -y$

(i) $x < y \Rightarrow y - x = z + y - z - x \in P \Rightarrow x + z < y + z$ (ii) $1 > 0 \Leftrightarrow 1 - 0 = 1 \in P$
(ii) Now $1 \neq 0$ since $a \neq 0 \Rightarrow a = a.1 \neq a.0 = 0$ Hence either $1 \in P$ or $-1 \in P$
But $-1 \in P \Rightarrow 1 = (-1)(-1) \in P$ contradiction to trichotomy. Hence $1 = 1 - 0 \in P \Rightarrow 1 > 0$
(iii) $x < y \Leftrightarrow y - x = -x - (-y) \in P \Leftrightarrow -y < -x$

(d) $A, B \subseteq \mathbf{R}$, $A, B \neq \emptyset$, $A \subseteq B$ and B is bounded above, show lubA $\leq$ lubB.

$\forall a \in A, a \in B$, $\forall b \in B$, $b \leq \text{lub}\, B \Rightarrow \forall a \in A$, $a \leq \text{lub}\, B$ so lubB an upper bound for A i.e. lubA $\leq$ lubB.

2. (a) Show from the definition of absolute value in the real numbers handout that $|x + y| \leq |x| + |y|$.
$x, y \geq 0$ or $x, y \leq 0$ $|x + y| = |x| + |y|$,
$x \geq 0, y < 0$ $|x + y| = |x| - |y| < |x| + |y|$ $\qquad x < 0, y \geq 0$ $|x + y| = -|x| + |y| < |x| + |y|$.

(b) Use (a) to prove by induction that $|x_1 + x_2 + \ \ldots\ + x_n| \leq |x_1| + |x_2| + \ \ldots\ + |x_n|$.

True for n = 2 by (a) So if $|x_1 + x_2 + \ \ldots\ + x_k| \leq |x_1| + |x_2| + \ \ldots\ + |x_k|$
$|x_1 + x_2 + \ \ldots\ + x_{k+1}| = |x_1 + x_2 + \ \ldots\ + x_k + x_{k+1}| \leq |x_1 + x_2 + \ \ldots\ + x_k| + |x_{k+1}| \leq |x_1| + |x_2| + \ \ldots\ + |x_k| + |x_{k+1}|$

(c) Show that the distance function $d(x, y) = |x - y|$ obeys the triangle law: $d(x, z) \leq d(x, y) + d(y, z)$.
$d(x, z) = |z - x| = |z - y + y - x| \leq |z - y| + |y - x| = d(x, y) + d(y, z)$

3. Find the least upper bound and greatest lower bound of each of the following subsets of **R** if they exist and determine if either of these is an element of the set concerned.

(i) $\mathbf{R} \setminus \{0\}$ (ii) $(-\infty, -1)$ (iii) $[-\pi, 2)$ (iv) $[-\pi, 2) \cap (\mathbf{R} \setminus \mathbf{Q})$ (v) $\left\{\frac{1}{n^2}{:}\ n \in \mathbf{N}\right\}$.

(i) No lub ($= \infty$) No glb ($= -\infty$) (ii) No glb ($= -\infty$) lun - -1 not in S.
(iii) glb = $-\pi$ in S lub = 2 not in S (iv) the same as $-\pi$ is in $\mathbf{R} \setminus \mathbf{Q}$. (v) glb = 0 not in S lub = 1 in S

4. For each of the following sequences, determine whether or not it is:

(a) convergent and if so find its limit,
(b) bounded and if so find a convergent subsequence
(c) find a subsequence which is increasing. or one which is decreasing, or both if possible.

(i) $\{1^n + (-1)^n : \ n \in \mathbf{N}\}$ (ii) $\left\{\frac{1}{n^2} : \ n \in \mathbf{N}\right\}$ (iii) $\left\{\frac{1^n}{n^2} + \frac{(-1)^n}{n^2} : \ n \in \mathbf{N}\right\}$ (iv) $\{n! : \ n \in \mathbf{N}\}$ (v) $\left\{\frac{n!}{n^n} : \ n \in \mathbf{N}\right\}$

(i) Not convergent as alternating between 2 and 0. Bounded. Absolute value bounded by 2. A convergent subsequence 2, 2, 2, 2 .... or 0, 0, 0, 0 ... hese are constant so are both increasing and decreasing.

(ii) $a_{n+1} - a_n = \frac{1}{(n+1)^2} - \frac{1}{n^2} = -\frac{2n+1}{n^2(n+1)^2} < 0$, $a_n \geq 0$ so monotone decreasing and bounded below. Hence convergent. The limit is zero $|a_n| = \left|\frac{1}{n^2}\right| < \varepsilon$ if $n > e^{-\frac{1}{2}}$. Bounded, decreasing and a;rady convergent.

(iii) This is basically a product of (i) and (ii) It is a product of a null and a bounded sequence so it is null i.e. bounded and convergent to 0. The odd terms form a decreasing subsequence $\left\{2, \frac{2}{9}, \frac{2}{25}, \ ...\right\}$.

(iv) Increasing unbounded, not convergent.

(v) $a_{n+1} - a_n = (n+1)!n^n - n!(n+1)^{n+1} = n!(n+1)\left(n^n - (n+1)^n\right) < 0$ so a_n decreasing bounded below by zero and hence convergent. Since $\frac{a_{n+1}}{a_n} = \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1$ the terms must get arbitrarily small so the limit is zero.

Prove each of the following specifically from the axioms given in the handout on real numbers.

(a) Given $a, b \in \mathbf{R}$, show there is a unique x such that $ax = b$.
There exists $y : ay = 1$ (multiplicative inverse). Let $x = by$ then $ax = aby = bay = b.1 = b$.
If $ax = b,\ ax' = b$ then $x - x' = 1(x - x') = ay(x - x') = ya(x - x') = yax - yax' = yb - yb = 0$, so $x = x'$.

If $x=b/a$ is defined to be $ax = b$, show:
(i) $a/b + c/d = (ad + bc)/bd$ if $b, d \neq 0$. (ii) $(a/b).(c/d) = ac/bd$ if $b, d \neq 0$.
(i) Let $a = x.b,\ c = y.d$, then $a/b + c/d = x + y$.
Now consider $z = (ad + bc)/bd = (xbd + byd)/bd = (x + y)bd/bd$, then $z(bd) = (x + y)bd$ so $z = x + y$.
(ii) $z = ac/bd = xbyd/bd = xybd/bd$, so $zbd = xybd$, and $z = xy$.

(b) (i) $x < y,\ y < z \Rightarrow x < z$ (ii) $x < y,\ z > 0 \Rightarrow xz < yz$.
(i) $x < y,\ y < z \Leftrightarrow y - x,\ z - y \in P \Rightarrow z - y + y - x = z - x \in P \Leftrightarrow x < z$.
(ii) $x < y,\ z > 0 \Leftrightarrow y - x,\ z \in P \Rightarrow z(y - x) = zy - zx \in P \Rightarrow xz < yz$.

(c) (i) $|xy| = |x|.|y|$ (ii) $||x| - |y|| \leq |x - y|$.
(i) $x.y \geq 0,\ |xy| = x.y = |x|.|y| = |-x|.|-y|,\ x.y < 0,\ |xy| = -x.y = |x|.|-y| = |-x|.|y|$.
(ii) $x, y \geq 0\ ||x| - |y|| = |x - y|,\ x, y < 0\ ||x| - |y|| = |-x - -y| = |x - y|$
$x \geq 0, y < 0\ ||x| - |y|| = |x + y| < |x + -y| = |x - y|$ since $x, -y$ both have the same sign but x, y have opposite sign. $x < 0, y \geq 0\ ||x| - |y|| = |-x + y| < |x - y|$ for the same reason.