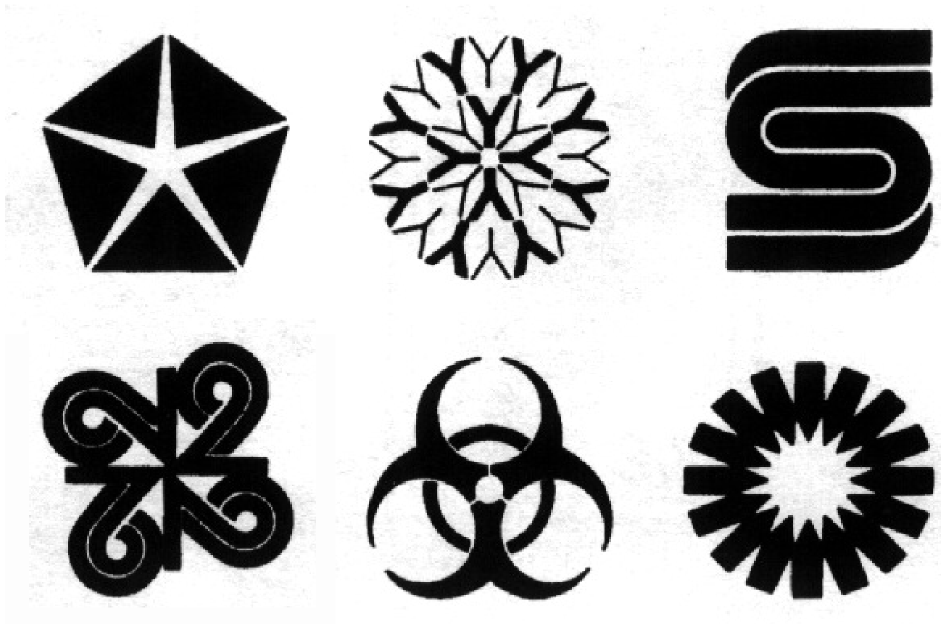
DEPARTMENT OF MATHEMATICS

MATHS 255FC — Assignment 7 solutions — Due 19th Sept 2001

1. For each design below determine the symmetry group.



Let D_n be the symmetry group of a regular n-gon and R_n be the subgroup of rotations. Then each of the figures above have a symmetry group either D_n or R_n depending on whether or not they are equivalent to their mirror image (e.g. by a rotation). Hence the symmetry groups of the figures are in order:
D_5, D_4, R_2, R_4, D_3, and D_{16}.

2. If G is a group, Show that $(a*b)^2 = a^2 * b^2 \Rightarrow a*b = b*a$.

$$a*b*a*b = a*a*b*b \Rightarrow a^{-1}*a*b*a*b*b^{-1} = a^{-1}*a*a*b*b*b^{-1}$$
$$\Rightarrow e*b*a*e = e*a*b*e \Rightarrow b*a = a*b$$

3. If G is a group for which every element $g \in G$ has $g^2 = e$, Show that G is commutative.

Consider any pair of elements $a,\ b \in G$, then
$$(a*b)^2 = e \Rightarrow a*b*a*b = e \Rightarrow a*a*b*a*b*b = a*e*b$$
$$\Rightarrow e*b*a*e = a*e*b \Rightarrow b*a = a*b$$

4. Let ***G*** be the group of matrices of each of the linear transformations corresponding to the group of symmetries of the square with vertices (±1,±1) under matrix multiplication. Show $G$is isomorphic to D_4.

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

This set of matrices generates the following cayley table by matrix multiplication

*	I	A	B	C	D	E	F	G
I	I	A	B	C	D	E	F	G
A	A	B	C	I	G	F	D	E
B	B	C	I	A	E	D	G	F
C	C	I	A	B	F	G	E	D
D	D	F	E	G	I	B	A	C
E	E	G	D	F	B	I	C	A
F	F	E	G	D	C	A	I	B
G	G	D	F	E	A	C	B	I

For example $E*C=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}=F$ **,** $F*E=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=A$

Notice that although the matrices operate on the left as functions they multiply in their usual order .
This is clearly isomorphic to D_4 as the Cayley tables as arranged have identical structures.

5. (a) Show that $\boldsymbol{Z}_9 \setminus \{0\}$ is not a multiplicative group.

Consider the partial Cayley table

*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	4	6	8	1	3	5	7
3	3	6	0	3	6	0	3	6
4	4							
5	5							
6	6							
7	7							
8	8							

The entries in the row determining the results of $3*g$ are inconsistent with this being a group for two reasons. Firstly it is not closed since $3*3 = 0$ is not in the set of elements, secondly $3*1 = 3*4 = 3*7$ viiolating the condensation law and its corollary that every element in any row must be distinct.

(b) Let $U(9) = \{\ \bar{n} : n\ \textit{is relatively prime to}\ 9\ \}$.
Show $(U(9), \bullet_9)$ is a multiplicative group, where $\bullet_9$ is multiplication modulo 9.

(c) Show *U(9)* is isomorphic to the additive group $\boldsymbol{Z}_6$.

*	2	4	5	7	8
1	2	4	5	7	8
2	4	8	1	5	7
4	8	7	2	1	5
5	1	2	7	8	4
7	5	1	8	4	2
8	7	5	4	2	1

*	1	2	4	8	7	5
1	1	2	4	8	7	5
2	2	4	8	7	5	1
4	4	8	7	5	1	2
8	8	7	5	1	2	4
7	7	5	1	2	4	8
5	5	1	2	4	8	7

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

The three Cayley tables above show (i) *U(9)* , (ii) *U(9)* with rows and columns rearranged, and (iii) $\boldsymbol{Z}_6$.
U(9) is clearly a group because it inherits associativity from integer addition, has an identity 1 and $5 = 2^{-1}$, $7 = 4^{-1}$, $8 = 8^{-1}$, so every element also has an inverse.

The above inverse pairings also suggest that this group may be isomorphic with $\boldsymbol{Z}_6$. Rearranging the Cayley table as in (ii) we can see how to define an isomorphism $f: U(9) \to Z_6$ as follows:
$f(1)=0,\ \ f(2)=1,\ \ f(4)=2,\ \ f(8)=3,\ \ f(7)=4,\ \ f(5)=5$.