DEPARTMENT OF MATHEMATICS

MATHS 255 Assignment 5 Solutions Due: 22 August, 2001

1. (6.3.2) A non-constant polynomial is *reducible* if it can be written as the product of two polynomials of smaller degree. Otherwise, it is *irreducible.*

Prove that every non-constant polynomial is irreducible or can be written as a product of non-constant irreducible polynomials.

We mimic the proof of the corresponding result for **N**. Let $S$ be the set of all exceptions, that is, $S$ is the set of all non-constant polynomials which are neither irreducible nor a product of non-constant irreducible polynomials. If $S$ is non-empty, then let $s \in S$ be a polynomial of smallest possible degree (It exists because the set of degrees of elements of $S$ is a non-empty subset of **N**, so it has a smallest element, say $d$ because **N** is well-ordered. Then let $s$ be one of the polynomials in $S$ which has degree $d$.) Since $s$ is not irreducible, it factors as a product $tu$ of non-constant polynomials of lesser degree. Then $t,u$ are not in $S$ (because their degree is too small) and hence they are products of non constant irreducible polynomials, and hence $s$ is too. This contradiction shows that $S$ is empty; and the proof is complete.

2. (7.2.7) Prove that the sum of the cubes of any three consecutive natural numbers is divisible by 9.

Let the three consecutive integers be $n-1,\ n,\ n+1$. The sum of their cubes simplifies to $3n(n^2+2)$. It therefore suffices to show that for all integers $n$, $n(n^2+2)$ is divisible by 3. Consider cases. If $n=3m$, then $n(n^2+2)$ is clearly divisible by 3. If $n=3m+1$ or $n=3m+2$, then $n^2+2=9m^2+6m+3$ or $9m^2+12m+6$, both of which are divisible by 3.

3. (7.2.19) Let $b,c$ be relatively prime integers. Show that for all integers $a$, $\gcd(a,b)$ and $\gcd(a,c)$ are relatively prime.

Let $d=\gcd(a,b)$, $e=\gcd(a,c)$. Suppose that $m>0$ and $m \mid d$ and $m \mid e$. Then $m$ divides $a, b, c,$ so that $m$ divides $b, c,$ and hence $m \mid \gcd(b,c)$. So $m \mid 1$, which means that $m=1$.

4. (7.3.3) Use the Euclidean Algorithm to find the greatest common divisor $d$ of 1452679 and 2306347 and find $u,v$ such that $d = 1452679\,u + 2306347\,v$

$d=1031$, $u=502$, $v=-797$. (other correct answers are possible.)